Complex Analysis Worksheet 2– Geometry of complex numbers

- As stated, one can treat complex numbers as points in the plane or as vectors in the plane.
- Geometrically, addition is just as vector addition from Calc 3 (tip-to-tail).
- One can view a complex number $z = a + bi$ in polar, just as in Calc 3. Here $r = |z|$ and $\theta = \arctan(\frac{b}{a})$ " + \pi" is the angle made with the $x$-axis. (We call the angle $\theta$ the argument of $z$, it is defined modulo $2\pi$, note this is usually written as $\theta = Arg(z)$).
- Recall that given modulus $|z|$ and angle $\theta$
  \[ z = |z| \cos \theta + |z| \sin \theta i = |z|(\cos \theta + i \sin \theta) \]

- **Multiplication in Polar**
  \[ w \cdot z = |w||z| \left[ \cos(\theta_w \cos \theta_z - \sin \theta_w \sin \theta_z + i(\cos \theta_w \sin \theta_z + \cos \theta_z \sin \theta_w) \right] \]
  \[ w \cdot z = |w||z| \left[ \cos(\theta_w + \theta_z) + i(\sin(\theta_w + \theta_z) \right] \]

  Lastly Note that
  \[ \left| \cos(\theta_w + \theta_z) + i(\sin(\theta_w + \theta_z) \right| = 1 \]

  Moral: Multiplying 2 complex numbers amounts to multiplying their norms and adding their polar angles.

  **Ex 1**: Use polar to find the following
  (a) $(1 + i) \cdot (-1 + \sqrt{3}i) = ?$
  (b) $(2 + 2i)^2 = ?$
  (c) $(-2 + 2i)^{40} = ?$

  - **Powers**: $z = |z|(\cos \theta + i \sin \theta)$
    \[ z^n = |z|^n \left( \cos(n\theta) + i \sin(n\theta) \right) \]

  - **De Moivre’s Theorem**: is the special case of the power formula with $|z| = 1$:
    \[ (\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta) \]
• **Division:** Since \( \frac{w}{z} = w \times \frac{1}{z} \) we can show that
  
  (a) \( \frac{1}{|z|} = \frac{1}{|z|} \) and (b) \( \theta \frac{1}{z} = -\theta_z + 2\pi k \) (or \( Arg(\frac{1}{z}) = Arg(z) + 2\pi k \))

So the quotient is:

\[
\frac{w}{z} = \frac{|w|}{|z|} \left[ \cos(\theta_w - \theta_z) + i \sin(\theta_w - \theta_z) \right]
\]

(quotients magnitude is the quotient of the magnitudes, and the angle of the quotient is the difference of the angles).

**Ex 2:** Use polar notation to find the following

(a) \((1 + i) ÷ (-1 + \sqrt{3}i) = ?\)

(b) \((2 + 2i) ÷ i = ?\)

(c) \(i ÷ (3 + 3i) = ?\)

**Ex 3:** Roots: Use the powers formula to find all possible values of \( \sqrt[4]{16i} \)

**General root formula:**

\[
\sqrt[n]{z} = \sqrt[n]{|z|} \left[ \cos\left(\frac{\theta + 2\pi k}{n}\right) + i \sin\left(\frac{\theta + 2\pi k}{n}\right) \right]
\]

**HW #2:**

1. Compute \((1 + \sqrt{3})^{110}\)

2. Pg 21: 23,24

3. Prove that \( Arg(\frac{1}{z}) = -Arg(z) + 2\pi k \)