Math 251 - Solution to worksheet on DE with Homogeneous coefficients (let $y = ux$) and Bernoulli’s

5. DE: $(y^2 + yx) \, dx - x^2 \, dy = 0$

$(u^2x^2 + uxx) \, dx = x^2(udx + xdu)$

$\frac{u^2x^2}{x} \, dx = \int \frac{1}{u^2} \, du$

$\ln x + C = -\frac{1}{u}$

$\ln x + C = -\frac{x}{y}$

General solution to DE is: $y = \frac{-x}{\ln x + C}$

7. DE: $(y - x) \, dx - (x + y) \, dy = 0$

$(ux - x) \, dx = (x + ux)(udx + xdu)$

$ux \, dx - x \, dx = ux \, dx + u^2 \, dx + x^2 \, du + xu \, du$

$-x \, dx - u^2 \, dx = x^2 \, du + x^2 \, du$

$-x(1 + u^2), \, dx = x^2(1 + u) \, du$

$-\int \frac{1}{x} \, dx = \int \frac{1 + u}{1 + u^2} \, du$

$C - \ln x = \arctan u + \frac{1}{2} \ln |1 + u^2|$

Implicit solution is: $C - \ln x = \arctan \left( \frac{y}{x} \right) + \frac{1}{2} \ln \left| 1 + \frac{y^2}{x^2} \right|$

11. DE: $(x^3 - y^3) \, dx + xy^2 \, dy = 0, \quad y(1) = 2$

$(x^3 - u^3 \, x^3) \, dx = -x(ux)^2(udx + xdu)$

$x^3 \, dx - u^3 \, x^3 \, dx = -x(ux)^2 \, u \, dx - x^4 \, u^2 \, du$

$-x^3 \, dx = x^4 \, u^2 \, du$

$-\int \frac{1}{x} \, dx = \int u^2 \, du$

$C - \ln x = \frac{y^3}{3}$

$\frac{y^3}{x^3} = 3C - 3 \ln x$

$y = \frac{x}{\sqrt[3]{3C - \ln x^3}}$
15. \[ xy' + y = y^{-2} \]

One see that \( n = -2 \) so that \( u = y^3 \) is the appropriate “substitution”.
Transform the DE to \( 3y^2 \frac{dy}{dx} + \frac{3}{x} y^3 = \frac{3y^2}{xy^2} \) which makes it easy to convert to \( \frac{du}{dx} + \frac{3}{x} u = \frac{3}{x} \). This is linear and solving yield \( x^3 u = x^3 + c \) Hence the solution to the DE is \( y = \sqrt[3]{1 + \frac{x}{x^2}} \)

17. \[ \frac{dy}{dx} = y(xy^3 - 1) \]

Rewriting as \( \frac{dy}{dx} + y = xy^4 \) makes it easier to see that \( n = 4 \) so that \( u = y^{1-4} = y^{-3} \) is the appropriate “substitution”. Multiply by \(-3y^{-4}\) on the DE gives: \(-3y^{-4} \frac{dy}{dx} - 3y^{-3} = -3x \) which makes it easy to convert to a linear DE in \( u \) and \( x \): \( \frac{du}{dx} - 3u = -3x \).

Solving yield \( e^{-3x} u = \int -3xe^{-3x} dx = x e^{-3x} + \frac{1}{3} e^{-3x} + c \) Therefore \( u = x + \frac{1}{3} + ce^{3x} \) Hence an implicit solution is \( y^{-3} = x + \frac{1}{3} + ce^{3x} \).

Working harder, we can convert this to \( y = \frac{3}{\sqrt[3]{3x + 1 + 3ce^{3x}}} \)

19. \[ t^2 \frac{dy}{dt} + y^2 = ty \]

Rewriting as \( \frac{dy}{dt} - \frac{1}{t^2} y = -\frac{1}{t^2} y^2 \) makes it easier to see that \( n = 2 \) so that \( u = y^{1-2} = y^{-1} \) is the appropriate “substitution”. Multiply by \(-y^{-2}\) on the DE gives: \(-y^{-2} \frac{dy}{dt} + \frac{1}{t} y^{-1} = \frac{1}{t^2} \) which makes it easy to convert to a linear DE in \( u \) and \( t \): \( \frac{du}{dt} + \frac{1}{t} u = \frac{1}{t^2} \).

Solving yield \( tu = \int \frac{1}{t} dt = C + \ln t \). Therefore \( u = \frac{C + \ln t}{t} \). Hence the solution is \( y = \frac{t}{C + \ln t} \) The book has it implicitly as \( e^{t/y} = ct \)