1. (15 points) Given that \( y''' + 9y'' + 27y' + 27y = f(x) \) has homogeneous (or complementary) solution given by \( y_c = c_1e^{-3x} + c_2xe^{-3x} + c_3x^2e^{-3x} \). Suppose the non-homogeneous part \( f(x) \) is given below, write down the form of particular solution (you do not need to find the constants)

(a) \( f(x) = 7 \cos x \)
(b) \( f(x) = 7 \sin 2x \)
(c) \( f(x) = (2x^2 + 3x) \cos x \)
(d) \( f(x) = (2x^2 + x)e^{-3x} \)
(e) \( f(x) = 2xe^{-3x} \sin x \)
(f) \( f(x) = \cos x + 5e^{-3x} \)
(g) \( f(x) = 7x^3 + \sin x \)
(h) \( f(x) = \frac{1}{7}x^4e^{-3x} \)

2. (15 points) Given the non-homogeneous DE \( y'' + y = g(x) \) has complementary solution given by \( y_c = c_1 \cos x + c_2 \sin x \). Suppose the non-homogeneous part \( g(x) \) is given below, write down the form of particular solution (not just the trial particular solution).

(a) \( g(x) = \cos 2x \)
(b) \( g(x) = 5e^{2x} \)
(c) \( g(x) = 7 \sin x \)
(d) \( g(x) = 3x \cos x \)
(e) \( g(x) = (2x^2 + 3x)e^{-3x} \)
(f) \( g(x) = e^x \sin x \)
(g) \( g(x) = x^3 + \sin x \)
(h) \( g(x) = x^2 \sin 2x - 2x \cos 2x \)

3. (15 points) Find the general solution of the following (homogeneous) DE

(a) \( y'' + 6y' + 9y = 0 \)  
(b) \( y'' + 2y' + 2y = 0 \)

4. (15 points) Find the general solution of the following (homogeneous) DE

(a) \( 4y'' + 20y' + 25y = 0 \)  
(b) \( y'' - 4y' + 8y = 0 \)

5. (15 points) Given that the complementary solution of \( y'' + 4y = 4x^2 \) is given by \( y_c = c_1 \cos 2x + c_2 \sin 2x \).

(a) Using the method of undetermined coefficients, find its particular solution
(b) Write down its general solution.
(c) Suppose \( y(0) = 0 \) and \( y'(0) = 2 \). Find the unique solution to the IVP.

6. (20 points) Test to see if the following are LI or LD.

(a) \( \sin 5x, \cos 5x, 5 \)  
(b) \( \sin^2 5x, \cos^2 5x, 5 \)

7. (10 points) Test to see if the following are linearly independent on \((0, \infty)\)

(a) \( x, x^2, 5x - 3x^2 \)  
(b) \( x, x - 1, x + 3 \)

8. (20 points) Test to see if the following are linearly independent on \((0, \infty)\)

(a) \( x, e^{2x}, xe^{2x} \)  
(b) \( e^x, e^{-2x}, xe^{-2x} \)

9. (20 points) Given the nonhomogenous differential equation \( y'' + 9y = 3 \tan 3x \). and given that \( y_1 = \cos 3x \) and \( y_2 = \sin 3x \) are two L.I. solution to \( y'' + 9y = 0 \), find a general solution to the given non-homog. DE by variation of parameters.
10. (20 points) Given the nonhomogenous differential equation \( y'' - y = 4e^{3x} \). Choosing \( y_1 = e^x \) and \( y_2 = e^{-x} \) in variation of parameters, we will have \( W = -2 \). Find a general solution to \( y'' - y = 4e^{3x} \) by variation of parameters.

11. (20 points) **Given the 3rd order DE \( y''' + y' = \tan x \) has fundamental set of solution given by \( \{1, \cos x, \sin x\} \). Supposed you assume that

\[
y_p(x) = u_1(x) + u_2(x) \cos x + u_3(x) \sin x
\]

Given that \( u_1(x) = \int \tan x \, dx \), find \( u_2(x) \) and \( u_3(x) \). \( \text{Hint: } W(x) = 1. \)

12. (20 points) **Given that the 3rd order DE \( y''' + 9y' = 81 \sec 3x \) has fundamental set of solution given by \( \{1, \cos 3x, \sin 3x\} \). Supposed \( y_p(x) = u_1(x) + u_2(x) \cos 3x + u_3(x) \sin 3x \)

Find \( u_1(x) \) and \( u_2(x) \). \( \text{Hint: } W(x) = 27. \)

13. (10 points) Given that \( y_1(x) = \frac{1}{x^2} \) is a solution to \( x^2 y'' + 6xy' + 6y = 0 \). Use the reduction of order to find the other independent solution.

14. (10 points) Given that \( y_1 = e^{5x} \) is a solution to \( y'' - 25y = 0 \). Use the reduction of order to find the other independent solution.

15. (10 points) Given that \( y_1 = x^4 \) is a solution to \( x^2 y'' - 7xy' + 16y = 0 \). Use the reduction of order to find the other independent solution.

16. (20 points) Given \( y''' + 4y' = 32 \sec 2x \) has fundamental set of solution given by \( \{1, \cos 2x, \sin 2x\} \). Supposed you assume that

\[
y_p(x) = u_1(x) + u_2(x) \cos 2x + u_3(x) \sin 2x,
\]

find \( u_1(x) \) and \( u_2(x) \). \( \text{Hint: } W(x) = 8. \)

17. (15 points) Given that the complementary solution of \( y'' - y' - 2y = 4x^2 - 4 \) is given by \( y_c = c_1 e^{2x} + c_2 e^x \).

(a) Using the method of undetermined coefficients, find its particular solution
(b) Write down its general solution.
(c) Suppose \( y(0) = 0 \) and \( y'(0) = 1 \). Find the unique solution to the IVP.

18. (15 points) Consider the RL circuit with a 5-ohm resistor and a 0.5-H inductor carries a current of 2 A at \( t = 0 \), at which time a voltage source \( E(t) = 3 \cos 120t \) V is added.

(a) Write down the initial value problem of the inductor current \( I \) for this circuit.
(b) Supposed the solution to your IVP above is given by

\[
I(t) = \frac{3}{725} (12 \sin 120t + \cos 120t) + \frac{1447}{725} e^{-10t},
\]

determine the inductor voltage at any time \( t \).
19. (15 points) An RC circuit has a 5-ohm resistor and a 0.0001 farad capacitor (which was initially discharged) with a voltage source of \( E(t) = 100 \) volt present at \( t = 0 \) sec.
   (a) Write down the initial value problem of \( Q(t) \) for this circuit.
   (b) Find the resistor voltage, \( E_R \) at any time \( t \).

20. (20 points) A brine solution of salt flows at a constant rate of 8L/min into a large tank that initially held 100L of brine solution in which was dissolved \( \frac{1}{2} \) kg of salt. The solution inside the tank is kept well stirred and flows out of the tank at the same rate. Assume the concentration of salt in the brine entering the tank is \( \frac{1}{20} \) kg/L.
   (a) Write down the initial value problem (IVP) of the amount of salt at any time.
      (\textit{do not use decimals, all numbers should be in reduced fraction.})
   (b) Find the unique solution of this IVP.

21. (10 points) A brine solution of salt flows at a constant rate of 8L/min into a large tank that initially held 100L of pure water. The solution inside the tank is kept well stirred and flows out of the tank at 6L/min. Assume the concentration of salt in the brine entering the tank is \( \frac{1}{20} \) kg/L.
   Write down the initial value problem (IVP) of the amount of salt at any time. (\textit{do not use decimals, all numbers should be in reduced fraction.})

22. (10 points) When a cake is removed from an oven, its temperature is measured at 300\(^\circ\)F. Three minutes later its temperature is 200\(^\circ\)F. How long will it take for the cake to cool off to a room temperature of 70\(^\circ\)F?
   (a) What is the differential equation for this model?
   (b) What is the general solution to the above DE? (\textit{do not find k.})