Answer to Worksheet–Optimization Business Application

1. An apartment complex has 250 apartments to rent. If they rent \( x \) apartments then their monthly profit, in dollars, is given by \( P = -8x^2 + 3200x - 80000 \).

   How many apartments should they rent in order to maximize their profit?

   \[
   \frac{dP}{dx} = -16x + 3200 = -16(x - 200). \text{ So } x = 200 \text{ is a critical value of } P.
   \]

   Also \( \frac{d^2P}{dx^2} = -16 \) which showed that \( P \) is a maximum when \( x = 200 \). Hence in order to maximize their profit, they should rent 200 apartments.

2. A production facility is capable of producing 60,000 widgets in a day and the total daily cost of producing \( x \) widgets in a day is given by,

   \[
   C = 250000 + 0.08x + \frac{200000000}{x}
   \]

   How many widgets per day should they produce in order to minimize production costs?

   \[
   \frac{dC}{dx} = 0.08 - \frac{200000000}{x^2} = \frac{0.08x^2 - 200000000}{x^2}. \text{ So if } \frac{dC}{dx} = 0, \ x^2 = 200,000,000/0.08 = 2500,000,000 \text{ and } x = 50,000 \text{ is a critical value of } C. \text{ Also } \frac{d^2C}{dx^2} = \frac{400000000}{x^3} \text{ which showed that } C \text{ is a minimum when } x = 50,000.
   \]

   Hence in order to minimize production costs, they should produce fifty thousand widgets per day.

3. The production costs per week for producing \( x \) widgets is given by,

   \[
   C = 2500 + 350x + 0.09x^2, \ 0 < x \leq 2000
   \]

   Find the number of widget production that will minimize the average cost per week.

   \[
   Answer: \text{ The average cost function is given by: } \frac{C}{x} = \frac{2500}{x} + 350 + 0.09x
   \]

   \[
   \Rightarrow \frac{dC}{dx} = \frac{-2500}{x^2} + 0.09. \text{ So if we set } \frac{dC}{dx} = 0 \text{ we get } x^2 = \frac{2500}{0.09} = \frac{250000}{9} \Rightarrow x = \frac{500}{3}. \text{ The second derivative of } \frac{C}{x} \text{ is positive for any } 0 < x \leq 2000. \text{ so the production level that will minimize the average cost per week is either 166 or 167. The value of } \frac{C}{x} \text{ is slightly higher when } x = 166.
   \]
4. The weekly cost to produce $x$ widgets is given by

$$C = 75,000 + 100x - 0.03x^2 + 0.000004x^3$$

and the demand function for the widgets is given by, $p(x) = 200 - 0.05x$, $0 \leq x \leq 10000$. Determine the number of widgets are sold that will maximize the total profit.

*Answer:* The total profit function is given by

$$P = R - C = xp - C = x(200 - 0.05x) - (75,000 + 100x - 0.03x^2 + 0.000004x^3) = 100x - 0.02x^2 - 75000 - 0.000004x^3$$

$$\Rightarrow P' = 100 - 0.04x - 0.000012x^2$$

Using quadratic formula to solve for the critical numbers yield $x = -5000, \frac{5000}{3}$. $P'' = -0.04 - 0.000024x$. so the second derivative is negative at $x = 1666.6667$ which proves that $P$ is a maximum at that value. So the number of widgets sold that will maximize the total profit will either be 1666 or 1667.

5. The monthly demand function and cost function for $x$ newspapers at a newsstand are $p = 5 - 0.001x$ and $C = 35 + 1.5x$. Find the number of newspaper sold that will make the profit function, $P$, a maximum. At what cost is the price of each newspaper that makes $P$ a maximum?

*Answer:* The total profit function is given by

$$P = R - C = xp - C = 5x - 0.001x^2 - (35 + 1.5x) = 3.5x - 0.001x^2 - 35$$

$$\Rightarrow P' = 3.5 - 0.002x$$

Critical Number: $\frac{3.5}{0.002} = 1750$. $P'' = -0.002$, so curve is always concave down.

Hence that critical number $x = 1750$ is a maximum.

the cost that makes $P$ a maximum is $p = 5 - 0.001(1750) = $3.25
6. A fast-food restaurant sells 20000 hamburgers per month if priced at $2. By increasing it by 50 cents, the sales goes down by 10000. Find the number of hamburgers and the price that has to be sold that maximizes the total revenue.

Answer: From the data point, we can deduce that slope of the demand function is 
\[ m = -\frac{0.5}{10000} = -\frac{1}{20000}. \]
Using the point slope equation, we get

\[ y - 2 = -\frac{1}{20000}(x - 20000) \]

solving for \( y \), which is also \( p \), we get the demand function 
\[ p = 3 - \frac{x}{20000} \]
The total revenue \( R = 3x - \frac{x^2}{20000} \)
Differentiating, we get \( R' = 3 - \frac{x}{10000} \) which gives a critical point of \( x = 30000 \) and \( p = 3 - 1.5 \) at that point. Hence the number of hamburger that maximizes the revenue is 30000 and the price that maximizes the revenue is $1.50.