MOST TRIANGLES ARE ACUTE

JOSEPH PREVITE AND MICHELLE PREVITE

In this paper, we deduce that most triangles in $n$ dimensional Euclidean space are acute for $n \geq 3$, unlike the case in the plane which was originally considered by Lewis Carroll in 1893 and further developed in this Magazine by Richard Guy in 1993 [2]. Further, we produce a formula that computes the ratio of obtuse triangles for all $n$ using generalized spherical coordinates.

1. Introduction and Background

Charles Lutwidge Dodgson whose "nom de plum" was Lewis Carroll first proposed the following problem in his work Pillow Problems Thought Out During Wakeful Hours in 1893:

"Three Points are taken at random on an infinite plane. Find the chance of their being the vertices of an obtuse-angled Triangle."

He argued that the ratio of obtuse triangle was $\frac{3\pi}{8\pi - 6\sqrt{3}} \approx 63.9\%$. However, Dodgson’s original question is somewhat ill-posed, since it is impossible to actually generate three random points on the plane where all points are equally likely. To be more precise, there is no translationally invariant probability measure on Euclidean space. Several authors have noted this and considered related problems on the unit disk, various rectangles, or where the coordinates of the triangle are chosen with respect to various probability distributions, all of which can be simulated (see [3]). Strang considers a related problem where the randomness is defined a bit differently [4].

For our purposes, we simply refine Dodgson’s ‘experiment’ more precisely by stating that after first fixing the longest side of the triangle to be $AB$, we generate a triangle by choosing one additional random point, where all such allowable points are equally likely. It turns out that (in Euclidean space) the resulting ratio of obtuse triangles is independent of the choice of $AB$. We will see that as the dimension increases, the likelihood of obtuse triangles decreases (to zero). Therefore, we make the loose assertion that there are more acute triangles than obtuse ones.

2. The planar case

We quickly revisit the argument in the plane produced by R. Guy, as we seek to generalize it to higher dimensions. Without loss of generality, we assume that the longest side of the triangle is the line segment from $A = (-D/2, 0)$ to $B = (D/2, 0)$ in the plane. Therefore, the third point must lie somewhere in the region $R$ formed by the intersection of two disks with radii $r$ centered at $A$ and $B$ respectively (see Figure).

Under the assumption that all such points are equally likely, one simply needs to find the area of this region and the area of the region that corresponds to all possible obtuse triangles.

\textit{Date: June 23, 2014.}
Figure 1. For $A = (-\frac{D}{2}, 0)$ and $B = (\frac{D}{2}, 0)$, the third point must be chosen in the intersection.

A nice precalculus exercise using the Pythagorean Theorem shows that the set of all points $(x, y)$ which make a right triangle with $\overline{AB}$ as hypotenuse is $x^2 + y^2 = (\frac{D}{2})^2$. Moreover, by the law of cosines, all points inside the circle create an obtuse triangle with longest side $\overline{AB}$, while all points chosen outside of this circle and inside the region $R$ form acute triangles with $\overline{AB}$.

If all such allowable points are equally likely, we simply need to compute the area of the inner circle and divide by the area of the full intersection. This then will be the probability of an obtuse triangle if the third point is chosen randomly (with all points equally likely) so that $\overline{AB}$ is the longest side.

The area of the inner circle is $\frac{1}{4}D^2\pi$. To compute the area of the intersection, we use polar coordinates with the origin relocated to $(-\frac{D}{2}, 0)$ to simplify the computation. We compute the area of the region indicated by the intersection in Figure 3. Note that $x = \frac{D}{2}$ becomes $r = \frac{D}{2} \sec \theta$ in polar. So,

$$2 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{D}{2} \sec \theta}^{D} r \, dr \, d\theta = D^2 (\frac{2}{3} \pi - \frac{\sqrt{3}}{2})$$

Taking the quotient, the probability of an obtuse triangle is

$$\frac{\frac{\pi}{\frac{3}{2} \pi - \frac{\sqrt{3}}{2}}}{\frac{3\pi}{8\pi - 6\sqrt{3}}} \approx 0.6393825609$$
Figure 2. For $A = (-\frac{D}{2}, 0)$ and $B = (\frac{D}{2}, 0)$, the third point $C$ inside the inner circle makes $\Delta ABC$ obtuse, while the third point outside the circle (but inside the intersection of the two larger circles) makes $\Delta ABC$ acute. Choosing $C$ on the inner circle produces a right triangle.

Figure 3. The area of the intersection in this figure can be computed in polar coordinates.
implying that in the plane there are more obtuse triangles than acute.

3. The case of $\mathbb{R}^3$

The situation is very much similar in higher dimensions. We handle the case of $\mathbb{R}^3$ separately before launching into higher dimensions. As before, we assume that the longest side of the triangle is positioned as the line segment from $A = (-\frac{D}{2}, 0, 0)$ to $B = (\frac{D}{2}, 0, 0)$. We then generate a third point that must be in the intersection of the two spheres of radius $D$ having centers $A$ and $B$.

One can again show via the Pythagorean Theorem that the choices of $C = (x, y, z)$ that form a right triangle having longest side $\overline{AB}$ is given by $x^2 + y^2 + z^2 = (\frac{D}{2})^2$. So the problem is completely similar to the two dimensional case, except that we now must compute volumes.

Once again, the sphere is simple, namely $V = \frac{4}{3}\pi (\frac{D}{2})^3 = \frac{1}{6}\pi D^3$. Next, we must compute the volume of the spherical cap (a spherical cap is the smaller region obtained by intersecting a plane with a sphere) indicated in Figure 6 using spherical coordinates. (Again, the region of intersection of these spheres is the union of 2 symmetric spherical caps).

This volume of this intersection is given by the spherical integral:
\[ \int_0^{2\pi} \int_0^\pi \int_0^{D/2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{5}{24} D^3 \pi. \]

So the volume associated with obtuse triangles divided by the allowable volume is

\[ \frac{1}{6} \pi D^3 = \frac{4}{10}. \]

We conclude that in three dimensions, it is more likely to have an acute triangle rather than an obtuse one.

4. The case of \( \mathbb{R}^n \)

As before, we can show that if the longest side of the triangle is positioned as the line segment from \( A = (-D/2, 0, \ldots, 0) \) to \( B = (D/2, 0, \ldots, 0) \), then the third point must be inside the intersection of the hyperspheres \( (x_1 - D/2)^2 + x_2^2 + \ldots + x_n^2 = D^2 \) and \( (x_1 + D/2)^2 + x_2^2 + \ldots + x_n^2 = D^2 \). Once again, the Pythagorean Theorem shows that the set of points \( (x_1, \ldots, x_n) \) which form a right triangle with hypotenuse \( AB \) is the hypersphere \( x_1^2 + x_2^2 + \ldots + x_n^2 = D^2 \).

As usual, points inside this hypersphere form obtuse triangles, while points outside form acute triangles.

Next, we introduce hyperspherical coordinates. These essentially generalize spherical coordinates to any dimension \( n \). For \( (x_1, \ldots, x_n) \in \mathbb{R}^n \), set

\[ \rho = \sqrt{\sum_{i=1}^n x_i^2} \]

and set

\[
\begin{align*}
    x_1 &= \rho \cos \phi_1 \\
    x_2 &= \rho \sin \phi_1 \cos \phi_2 \\
    x_3 &= \rho \sin \phi_1 \sin \phi_2 \cos \phi_3 \\
    &\vdots \\
    x_{n-1} &= \rho \sin \phi_1 \sin \phi_2 \ldots \sin \phi_{n-2} \cos \phi_{n-1} \\
    x_n &= \rho \sin \phi_1 \sin \phi_2 \ldots \sin \phi_{n-2} \sin \phi_{n-1}
\end{align*}
\]

where \( 0 \leq \phi_{n-1} < 2\pi \) and \( 0 \leq \phi_j \leq \pi \) for \( 1 \leq j \leq n-2 \).

Computing the determinant of the Jacobian transforms the volume element

\[ dx_1 \, dx_2 \ldots \, dx_n = \rho^{n-1} \sin^{n-2}(\phi_1) \sin^{n-3}(\phi_2) \ldots \sin(\phi_{n-2}) d\rho \, d\phi_1 \ldots \, d\phi_{n-1}. \]

Armed with this, we can compute the volumes of the hypersphere that corresponds to obtuse triangles and of the larger region consisting of two symmetric hyperspherical caps.

\[
\int_0^{2\pi} \int_0^\pi \ldots \int_0^\pi \int_0^D \rho^{n-1} \sin^{n-2}(\phi_1) \sin^{n-3}(\phi_2) \ldots \sin(\phi_{n-2}) d\rho \, d\phi_1 \ldots \, d\phi_{n-1} \]

\[= \frac{D^n}{n \cdot 2^n} \left( \int_0^\pi \sin^{n-2}(\phi_1) \, d\phi_1 \right) \left( \int_0^\pi \sin^{n-3}(\phi_2) \, d\phi_2 \right) \ldots \left( \int_0^{2\pi} \sin(\phi_{n-1}) \, d\phi_{n-1} \right) \]
We also can compute the volume of the (shifted) hyperspherical cap (which is half of the required volume). Since \( x_1 = \frac{D}{2} \) transforms, as before, to \( \rho = \frac{D}{2 \cos \phi_1} \) we obtain:

\[
\int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \int_0^{D/2 \sec \phi_1} \rho^{n-1} \sin^{n-2}(\phi_1) \sin^{n-3}(\phi_2) \cdots \sin(\phi_{n-2}) d\rho d\phi_1 d\phi_2 \cdots d\phi_{n-2} d\phi_{n-1}
\]

\[
= \left( \int_0^\pi \sin^{n-3}(\phi_2) d\phi_2 \right) \cdots \left( \int_0^{2\pi} \sin(\phi_{n-1}) d\phi_{n-1} \right) \left[ \int_0^{\pi/2} \int_0^D \rho^{n-1} \sin^{n-2}(\phi_1) d\rho d\phi_1 \right]
\]

\[
= \left( \int_0^\pi \sin^{n-3}(\phi_2) d\phi_2 \right) \cdots \left( \int_0^{2\pi} \sin(\phi_{n-1}) d\phi_{n-1} \right) \left[ \int_0^{\pi/2} \int_0^D \rho^{n-1} d\rho d\phi_1 \right] \left[ \int_0^{\pi/2} \frac{D^n}{n} \left( 1 - \frac{1}{2^n \sec^n(\phi_1)} \right) \sin^{n-2}(\phi_1) d\phi_1 \right]
\]

Taking the ratio of the volumes and simplifying, (remembering to double the denominator) we obtain:

\[
\frac{\frac{1}{2^n} \left( \int_0^\pi \sin^{n-2}(\phi_1) d\phi_1 \right)}{2 \int_0^{\pi/2} \left( 1 - \frac{1}{2^n \sec^n(\phi_1)} \right) \sin^{n-2}(\phi_1) d\phi_1}
\]

The integral \( \int_0^{\pi/2} \sec^n(\phi_1) \sin^{n-2}(\phi_1) d\phi_1 \) is a straightforward u-substitution setting \( u = \tan(\phi_1) \) yielding \( \frac{(\sqrt{3})^{n-1}}{n-1} \).

The other integrals are powers of sine, for which we can turn to the familiar recursive formula:

\[
\int_a^b \sin^n(x) \, dx = -\frac{1}{n} \sin^{n-1}(x) \cos(x) \bigg|_a^b + \left( \frac{n-1}{n} \right) \int_a^b \sin^{n-2}(x) \, dx
\]

The following table shows ratio of obtuse triangles to allowable triangles for \( 2 \leq n \leq 10 \):
We next show that the ratio
\[
\frac{\frac{1}{2^n} \left( \int_0^\pi \sin^{n-2}(\phi_1) \, d\phi_1 \right)}{2 \int_0^{\pi/3} \left( 1 - \frac{1}{2^n} \sec^n(\phi_1) \right) \sin^{n-2}(\phi_1) \, d\phi_1}
\]
tends to zero as \( n \to \infty \).

To see this, simply note that
\[
\frac{\frac{1}{2^n} \left( \int_0^\pi \sin^{n-2}(\phi_1) \, d\phi_1 \right)}{2 \int_0^{\pi/3} \left( 1 - \frac{1}{2^n} \sec^n(\phi_1) \right) \sin^{n-2}(\phi_1) \, d\phi_1} \leq \frac{1}{2^n \pi} \left( \int_0^\pi \cos(\phi_1) - \frac{1}{2^n} \sec^n(\phi_1) \right) \sin^{n-2}(\phi_1) \, d\phi_1
\]
This resolves to
\[
\frac{\pi(n - 1)}{2(\sqrt{3})^{n-1}},
\]
which tends to zero as \( n \to \infty \).
5. Conclusion, Final Remarks, Open Problems

We have shown that as the dimension increases, the ratio of obtuse triangles to total allowable triangles tends to zero. In essence, obtuse triangles in large dimension Euclidean space are a rarity. Already in 3 dimensions, there is a clear minority of obtuse triangles, which is very unlike the case in the plane.

One could take this work forward in several directions. Namely, fix a geodesic segment on an $n$ dimensional sphere having fixed length. Of all allowable points on the sphere which form a 'triangle' with longest length given by our specified geodesic segment, what is the probability that an obtuse triangle would be created? What about on the hyperbolic disk?

References