3.7 Wronskians and Variation of Parameters

In this section, we will need the concept of the Wronskian defined below:

### 3.7.1 The Wronskian

Define the Wronskian of two differentiable functions \( f(t) \) and \( g(t) \) to be

\[
W[f, g](t) = \begin{vmatrix}
  f(t) & g(t) \\
  f'(t) & g'(t)
\end{vmatrix} = f(t)g'(t) - g(t)f'(t)
\]

**Example 3.27** Compute the Wronksian of \( f(t) = t^2 \) and \( g(t) = \sin(6t) \)

Solution:

\[
W[f, g](t) = \begin{vmatrix}
  f(t) & g(t) \\
  f'(t) & g'(t)
\end{vmatrix} = f(t)g'(t) - g(t)f'(t),
\]

so

\[
W[t^2, \sin(6t)] = \begin{vmatrix}
  t^2 & \sin(6t) \\
  2t & 6\cos(6t)
\end{vmatrix}
= 6t^2\cos(6t) - 2t\sin(6t).
\]

**Example 3.28** Compute the Wronksian of \( f(t) = \sqrt{t} \) and \( g(t) = e^{2t} \)

Solution:

\[
W[f, g](t) = \begin{vmatrix}
  f(t) & g(t) \\
  f'(t) & g'(t)
\end{vmatrix} = f(t)g'(t) - g(t)f'(t),
\]

so

\[
W[\sqrt{t}, e^{2t}] = \begin{vmatrix}
  \sqrt{t} & e^{2t} \\
  \frac{1}{2\sqrt{t}} & 2e^{2t}
\end{vmatrix}
= 2\sqrt{t}e^{2t} - \frac{1}{2\sqrt{t}}e^{2t}.
\]

Notice that the Wronskian of two functions is again a new function whose domain depends upon the domains of \( f \) and \( g \) and their derivatives as illustrated in the previous example. There are some properties of the Wronskian, which are straightforward to verify.
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1. \( W[0, g(t)] = 0 \)

2. \( W[f(t), f(t)] = 0 \)

3. \( W[1, g(t)] = g'(t) \)

4. \( W[f(t), g(t) + h(t)] = W[f(t), g(t)] + W[f(t), h(t)] \)

5. \( W'[f(t), g(t)] = f g'' - f'' g = W[f, g'] + W[f', g] \)

6. For any constant \( c \),
   \[
   W[f(t), cg(t)] = cW[f(t), g(t)] = W[cf(t), g(t)]
   \]

7. \( W[f(t), g(t)] = -W[g(t), f(t)] \)

The last property tells us that the order of the functions in the Wronskian is important.

3.7.2 Variation of Parameters

Suppose that
\[
y''(t) + a(t)y'(t) + b(t)y(t) = F(t)
\]
is a second order linear ODE and that \( c_1 y_1(t) + c_2 y_2(t) \) is a general solution to the associated homogeneous DE. Then
Suppose that
\[ y''(t) + a(t)y'(t) + b(t)y(t) = F(t) \]
is a second order linear ODE and that \( c_1y_1(t) + c_2y_2(t) \) is a general solution to the associated homogeneous DE. Then
\[ y_P(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \]
is a particular solution to the nonhomogeneous DE, where
\[
\begin{align*}
v_1(t) &= -\int \frac{F(t)y_2(t)}{W[y_1, y_2]} \, dt \\
v_2(t) &= \int \frac{F(t)y_1(t)}{W[y_1, y_2]} \, dt
\end{align*}
\]
(As usual, the antiderivatives in the formulas for \( v_1, v_2 \) denote any one antiderivative.)

**Example 3.29** Find a solution to the nonhomogeneous DE
\[ y'' + 4y = \sec(2t) \]
\[ \textbf{Solution:} \] Note that the general solution to the homogeneous DE is
\[ y_{\text{homo}}(t) = c_1 \cos(2t) + c_2 \sin(2t), \]
so \( y_1 = \cos(2t) \) and \( y_2 = \sin(2t) \).
Next, note that
\[
W[y_1, y_2] = W[\cos(2t), \sin(2t)] = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2 \sin(2t) & 2 \cos(2t) \end{vmatrix} = 2 \cos^2(2t) + 2 \sin^2(2t) = 2.
\]
So
\[
v_1(t) = -\int \frac{\sec(2t) \sin(2t)}{2} \, dt = -\int \frac{\tan(2t)}{2} \, dt = \frac{1}{4} \ln |\sin(2t)|
\]
and
\[
v_2(t) = \int \frac{\sec(2t) \cos(2t)}{2} \, dt = \int \frac{1}{2} \, dt = \frac{1}{2} t
\]
Thus \( y_p(t) = v_1 y_1 + v_2 y_2 = \frac{1}{4} \ln |\sin(2t)| \cos(2t) + \frac{1}{2} t \sin(2t) \) solves the nonhomogenous DE.

Note that the general solution to the DE (by the Superposition Principle) is \( y(t) = \frac{1}{4} \ln |\sin(2t)| \cos(2t) + \frac{1}{2} t \sin(2t) + c_1 \cos(2t) + c_2 \sin(2t) \).

The next example illustrates that this method works even when the coefficients are not all constant (unlike the method of undetermined coefficients). Note in the example that the DE needs to be put into standard form before we can use the formula.

**Example 3.30** Find a solution to the nonhomogeneous DE

\[
t^2 y'' - 2y = \frac{3}{t}, \quad t \neq 0
\]

given that \( y = (t)c_1 t^2 + c_2 \frac{1}{t} \) solves

\[
t^2 y'' - 2y = 0
\]

**Solution:** Note that the general solution to the homogeneous DE is

\[
y_{hom} (t) = c_1 t^2 + c_2 \frac{1}{t}
\]

so \( y_1 = t^2 \) and \( y_2 = \frac{1}{t} \). Further note that in standard form, the DE becomes

\[
y'' - \frac{2}{t^2} y = \frac{3}{t^3}, \quad t \neq 0
\]

Next, note that

\[
W[y_1, y_2] = W[t^2, \frac{1}{t}] = \left| \begin{array}{cc} t^2 & \frac{1}{t} \\ 2t & -\frac{1}{t^2} \end{array} \right| = -1 - 2 = -3.
\]

So

\[
v_1(t) = -\int \frac{3}{t^3} \, dt = \int \frac{1}{t^4} \, dt = -\frac{1}{3} \frac{1}{t^3}
\]

and

\[
v_2(t) = \int \frac{3}{t^3} \, dt = -\int \frac{1}{t} \, dt = -\ln |t|
\]
Thus \( y_P(t) = v_1y_1 + v_2y_2 = -\frac{1}{3t^3} - \frac{\ln|t|}{t} \) solves the nonhomogenous DE. □

As one might guess, the previous examples were chosen very carefully so that the antiderivatives could be computed in closed form. At first, it might seem that this method allows us only to solve a small set of problems, in particular, problems where the antiderivatives in the formula are computable. However, this method becomes extremely powerful and versatile if we recall that the antiderivatives of \( G(t) \) are simply obtained by \( \int_{t_0}^{t} G(w) \, dw \), where \((t_0, t)\) is in the domain of \( G \). Hence, the variation of parameters method allows us to obtain a particular solution even when the antiderivatives do not “work out nicely”. The tradeoff is that one may need to approximate a definite integral to evaluate a solution as in the next example.

**Example 3.31** Find the solution to the nonhomogeneous ODE/IVP

\[ y'' + 4y = \ln(t + 1), \quad y(0) = 0, \quad y'(0) = 1 \]

and use it to approximate \( y(2) \)

**Solution:** As in the previous example, the general solution to the homogeneous DE is

\[ y_{homo}(t) = c_1 \cos(2t) + c_2 \sin(2t), \]

so \( y_1 = \cos(2t) \) and \( y_2 = \sin(2t) \) and

\[
W[y_1, y_2] = W[\cos(2t), \sin(2t)] = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2\cos^2(2t) + 2\sin^2(2t) = 2.
\]

So

\[
v_1(t) = - \int \frac{\ln(t) \sin(2t)}{2} \, dt = -\frac{1}{2} \int_0^t \ln(w + 1) \sin(2w) \, dw
\]

and

\[
v_2(t) = \int \frac{\ln(t) \cos(2t)}{2} \, dt = \frac{1}{2} \int_0^t \ln(w + 1) \cos(2w) \, dw
\]

(we take \( t_0 = 0 \)).
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So the general solution to the nonhomogeneous DE is

\[ y(t) = c_1 \cos(2t) + c_2 \sin(2t) \]

\[ - \cos(2t) \left( \frac{1}{2} \int_0^t \ln(w+1) \sin(2w) \, dw \right) + \sin(2t) \left( \frac{1}{2} \int_0^t \ln(w+1) \cos(2w) \, dw \right) \]

Since \( y(0) = 0 \) we have

\[ 0 = c_1 - \left( \frac{1}{2} \int_0^0 \ln(w+1) \sin(2w) \, dw \right) \]

so \( c_1 = 0 \)

Using the product rule and the second fundamental theorem of calculus,

\[ y'(t) = 2c_2 \cos(2t) - \cos(2t) \left( \frac{1}{2} \ln(t+1) \sin(2t) \right) \]

\[ + 2 \sin(2t) \left( \frac{1}{2} \int_0^t \ln(w+1) \sin(2w) \, dw \right) + 2 \cos(2t) \left( \frac{1}{2} \int_0^t \ln(w+1) \cos(2w) \, dw \right) \]

\[ + \sin(2t) \left( \frac{1}{2} \ln(t+1) \cos(2t) \right) \]

Plugging in \( t = 0 \) yields \( 1 = 2c_2 \) so \( c_2 = \frac{1}{2} \).

So the particular solution we seek is

\[ y(t) = \frac{1}{2} \sin(2t) - \cos(2t) \left( \frac{1}{2} \int_0^t \ln(w+1) \sin(2w) \, dw \right) + \sin(2t) \left( \frac{1}{2} \int_0^t \ln(w+1) \cos(2w) \, dw \right) \]

\[ y(2) = \frac{1}{2} \sin(4) - \cos(4) \left( \frac{1}{2} \int_0^2 \ln(w+1) \sin(2w) \, dw \right) + \sin(4) \left( \frac{1}{2} \int_0^2 \ln(w+1) \cos(2w) \, dw \right) \approx .0014974951 \]

Above, we obtained an approximation of \( y(2) \) by approximating the definite integrals (you could approximate the integrals Simpson’s method, the Trapezoidal method, Upper/Lower sums, or simply a calculator that can approximate definite integrals).
Exercises

Use the variation of parameters method to find a particular solution to the DE

1. $y'' + 3y' + 2y = 4e^t$

2. $y'' + 3y' + 2y = t$

3. $y'' + y = \tan t$

4. $y'' + y = \csc t$

5. $y'' + 4y = \sin(2t) \cos(2t)$

Use the variation of parameters method to find a general solution to the DE

6. $y'' + 9y = \cot(3t)$

7. $y'' + y = \csc t$

8. $y'' + 4y = \sin(2t) \cos(2t)$

9. $t^2y'' - 6y = t^4$ given that $y(t) = c_1 t^3 + c_2 \frac{1}{t^2}$ solve the homogeneous DE. (Hint: Put the DE in standard form first!)

Use the variation of parameters method to approximate the particular value solution to the ODE/IVP, you will need to approximate some definite integrals

10. $y'' + 4y' + 3y = \sqrt{t}, \ y(1) = 1, \ y'(1) = 2$

11. $y'' + 9y = \frac{1}{t}, \ y(1) = 0, \ y'(1) = 2$