3.4 Application-Spring Mass Systems (Unforced and frictionless systems)

Second order differential equations arise naturally when the second derivative of a quantity is known. For example, in many applications the acceleration of an object is known by some physical laws like Newton’s Second Law of Motion $F = ma$. One particularly nice application of second order differential equations with constant coefficients is the model of a spring mass system.

Suppose that a mass of $m$ kg is attached to a spring. From physics, Hooke’s Law states that if a spring is displaced a distance of $y$ from its equilibrium position, then the force exerted by the spring is a constant $k > 0$ multiplied by the displacement of the $y$. In other words,

$$F_{spring} = -ky.$$ 

The negative sign above is due to the fact that the force will always be in the opposite direction of the displacement.

3.4.1 Undamped Springs (no friction)

We are now in a position to formulate a model of a spring/mass system. By Newton’s Second Law,

$$F = ma$$

and we realize that $a = y''(t)$. So we obtain the second order differential equation

$$my'' = -ky$$

which we rewrite as

$$my'' + ky = 0$$

where $m > 0$ and $k > 0$.

Of course we can solve this system for all values of $m, k$ since it is a homogeneous linear second order DE with constant coefficients.

It has general solution:

$$y(t) = c_1 \cos(\sqrt{\frac{k}{m}} t) + c_2 \sin(\sqrt{\frac{k}{m}} t)$$
Figure 3.1: A spring mass system
3.4. APPLICATION-SPRING MASS SYSTEMS (UNFORCED AND FRICTIONLESS SYSTEMS)

The long term behavior of this spring/mass system as suggested from the
general solution above is that the mass will oscillate forever, which is not
realistic. This suggests that our model is missing some key physical feature.
Indeed, we have neglected frictional forces. However, if it were possible to
have no friction, then the model reflects what we would expect.

Example 3.15 A spring with spring constant 18N/m is attached to a 2kg
mass with negligible friction. Determine the period that the spring mass
system will oscillate for any non-zero initial conditions.

Solution: From above, we have a spring mass system modelled by the DE

\[ 2y'' + 18y = 0 \]

which has general solution given by

\[ y(t) = c_1 \cos(\sqrt{\frac{18}{2}}t) + c_2 \sin(\sqrt{\frac{18}{2}}t) = c_1 \cos(3t) + c_2 \sin(3t) \]

Since period of \( \cos t \) is \( 2\pi \), then the period of \( \cos(3t) \) and \( \sin(3t) \) is \( \frac{2\pi}{3} \). Therefore, the period of \( c_1 \cos(3t) + c_2 \sin(3t) \) is also \( \frac{2\pi}{3} \). □

Note: The frequency of \( \cos(\beta t) \) is often defined two (different) ways, one
way is \( \text{frequency} = \frac{1}{\text{period}} = \frac{\beta}{2\pi} \). Another similar definition is the angular
frequency of \( \cos(\beta t) \) which is simply \( \beta \). We suggest avoiding frequencies al-
together and working with the period, since, then, there is no confusion.

3.4.2 Converting \( c_1 \cos(\beta t) + c_2 \sin(\beta t) \) into phasor form

\( A \cos(\beta t - \phi) \)

In this section we show how to convert

\[ y = c_1 \cos(\beta t) + c_2 \sin(\beta t) \tag{3.12} \]

into the form \( y = A \cos(\beta t - \phi) \).

Using the difference formulas from trigonometry

\[ A \cos(X - \phi) = A \cos \phi \cos X + A \sin \phi \sin X \]
and taking \( X = \beta t \) in this formula, and matching with formula (3.13) we obtain, \( c_1 = A \cos \phi \) and \( c_2 = A \sin \phi \).

Note that
\[ c_1^2 + c_2^2 = A^2(\cos^2 \phi + \sin^2 \phi) = A^2 \]
so
\[ A = \sqrt{c_1^2 + c_2^2}. \]

So long \( c_1 \neq 0 \), we have
\[ \frac{c_2}{c_1} = \frac{A \sin \phi}{A \cos \phi} = \tan \phi \]
and so Note that \( \phi = \arctan\left(\frac{c_2}{c_1}\right) \) if \( c_1 > 0 \) and \( \phi = \arctan\left(\frac{c_2}{c_1}\right) + \pi \) if \( c_1 < 0 \).

You may realize that the values of \( A \) and \( \phi \) are simply the polar coordinates \((r, \theta)\) of the point \((c_1, c_2)\).

In more compact form, so long as \( c_1 \neq 0 \)
\[ \phi = \arctan\left(\frac{c_2}{c_1}\right) + \frac{\pi}{2} \left(1 - \frac{c_1}{|c_1|}\right) \]
Note that if \( c_1 = 0 \) then (from polar coordinates) we see that \( A = |c_2| \) and \( \phi = \frac{\pi}{2} \) if \( c_2 > 0 \) and \( \phi = -\frac{\pi}{2} \) if \( c_2 < 0 \).

We summarize below:

<table>
<thead>
<tr>
<th>Phasor Form</th>
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<tbody>
<tr>
<td>To convert ( y = c_1 \cos(\beta t) + c_2 \sin(\beta t) ) into the form ( y = A \cos(\beta t - \phi) ) :</td>
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<tr>
<td>( A = \sqrt{c_1^2 + c_2^2} )</td>
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<tr>
<td>and ( \phi = \begin{cases} \arctan\left(\frac{c_2}{c_1}\right) + \frac{\pi}{2} \left(1 - \frac{c_1}{</td>
</tr>
<tr>
<td>Moreover, ((A, \phi)) are the polar coordinates of the rectangular point ((c_1, c_2)).</td>
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Note: In the above formula, \( c_1 \) must always be the coefficient of the sine term and \( c_2 \) must be the coefficient of the cosine term. Also, the sine and cosine functions must have the same argument.
Example 3.16 Convert each of the following to phase angle (phasor) notation.

(a) \[ y = 4 \cos(3t) - 4 \sin(3t) \]

(b) \[ y = -\cos(2t) + \sqrt{3} \sin(2t) \]

(c) \[ y = -7 \sin(t) \]

Solution:
(a) We see that \( c_1 = 4 \) and \( c_2 = -4 \) so
\[
A = \sqrt{c_1^2 + c_2^2} = \sqrt{4^2 + (-4)^2} = \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2}
\]
\[
\phi = \arctan \left( \frac{-4}{4} \right) + \frac{\pi}{2} \left( 1 - \frac{4}{|-4|} \right) = \arctan(-1) + 0 = -\frac{\pi}{4}
\]
So
\[ y = 4\sqrt{2} \cos(3t - \frac{\pi}{4}) \]

(b) We see that \( c_1 = -1 \) and \( c_2 = \sqrt{3} \) so
\[
A = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2
\]
\[
\phi = \arctan \left( \frac{\sqrt{3}}{-1} \right) + \frac{\pi}{2} \left( 1 - \frac{-1}{|-1|} \right) = \arctan(-\sqrt{3}) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}
\]
So
\[ y = 2 \cos(2t + \frac{2\pi}{3}) \]

(c) We see that \( c_1 = 0 \) and \( c_2 = -7 \). So we cannot use the formula involving arc tangent. But when we plot the point \((0, -7)\) we see that in polar it is \( r = 7 \) and \( \theta = \frac{3\pi}{2} \) so \( A = 7 \) and \( \phi = \frac{3\pi}{2} \) and
\[ y = 7 \cos(t + \frac{3\pi}{2}) \]
Example 3.17  A spring with spring constant 4 N/m is attached to a 1 kg mass with negligible friction. If the mass is initially displaced to the right of equilibrium by 0.5 m and has an initial velocity of 1 m/s toward equilibrium. Compute the amplitude of the oscillation.

Solution:
As before, the spring mass system corresponds to the DE
\[ y'' + 4y = 0. \]

Since the mass is displaced to the right of equilibrium by 0.5 m, we have \( y(0) = \frac{1}{2} \). Since the mass an initial velocity of 1 m/s toward equilibrium (to the left) \( y'(0) = -1 \).

Solving the spring mass system, we obtain the general solution
\[ y(t) = c_1 \cos(2t) + c_2 \sin(2t). \]

\[ y(0) = \frac{1}{2} \] gives \( c_1 = \frac{1}{2} \) and since
\[ y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t), \]
we see that \( y'(0) = -1 \) implies that
\[ -1 = 2c_2 \]
or \( c_2 = -\frac{1}{2} \).
So
\[ y(t) = \frac{1}{2} \cos(2t) - \frac{1}{2} \sin(2t). \]

Converting to phase/angle notation, we see \( A = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2} \) so the amplitude of oscillation will be \( \frac{\sqrt{2}}{2} m \).

Note that \( \phi = -\frac{\pi}{4} \) since the polar angle of \( \left(\frac{1}{2}, -\frac{1}{2}\right) \) is \( -\frac{\pi}{4} \). □

Exercises

Convert to phase/angle (phasor) form

1. \( y = 2 \cos t - 2 \sin t \)
2. \( y = -4 \cos(6t) - 4 \sin(6t) \)

3. \( y = \cos(4t) \)

4. \( y = -12 \sin(\sqrt{2}t) \)

5. A spring with spring constant \( 2N/m \) is attached to a 1kg mass with negligible friction. Compute the period of the oscillation for any non-zero initial conditions.

6. A spring with spring constant \( 16N/m \) is attached to a 1kg mass with negligible friction. If the mass is initially displaced to the left of equilibrium by \( 0.25m \) and has an initial velocity of \( 1 \text{ m/s} \) toward equilibrium. Compute the amplitude and period of the oscillation.

7. A spring with spring constant \( 16N/m \) is attached to a 1kg mass with negligible friction. If the mass is initially at equilibrium with an initial velocity of \( 2 \text{ m/s} \) toward the left. Compute the amplitude and period of the oscillation.

8. A spring with spring constant \( 2N/m \) is attached to a 1kg mass with negligible friction. If the mass is initially 1m to the left of equilibrium with no initial velocity. Compute the amplitude and period of the oscillation.