Derivation of the Governing Differential Equations

- **Assumption 1:**
  The flow between Arteries and Veins $Q_{AV}$ is proportional to the pressure difference.
  
  \[ Q_{AV} = \frac{P_A - P_V}{R} \]
  
  In this equation, $R$ is called the resistance to flow.

- **Assumption 2:**
  Volume changes are proportional to the pressure changes:
  
  \[ \frac{dV_A}{dt} = C_A \frac{dP_A}{dt} \quad (1) \]
  
  \[ \frac{dV_V}{dt} = C_V \frac{dP_V}{dt} \quad (2) \]

  In these equations, $C_A$ and $C_V$ are the compliances.

- **Conservation of mass in each compartment**
  
  Flow Rate in – Flow Rate out = rate of volume change \( (4) \)

  \[ \text{compartment A:} \quad Q_{HA}(t) - Q_{AV} = \frac{dV_A}{dt} \quad (5) \]

  \[ \text{compartment V:} \quad Q_{AV} - Q_{VH}(t) = \frac{dV_V}{dt} \quad (6) \]

  \[ \text{compartment A:} \quad Q_{HA}(t) - \frac{P_A - P_V}{R} = C_A \frac{dP_A}{dt} \quad (7) \]

  \[ \text{compartment V:} \quad \frac{P_A - P_V}{R} - Q_{VH}(t) = C_V \frac{dP_V}{dt} \quad (8) \]
The model for Arterial pressure $P_A$ and Venous pressure $P_V$ in mmHg, results in the following system of differential equations for pressures.

\[
\frac{dP_A}{dt} = \frac{1}{C_A} \left( -\frac{1}{R} P_A + \frac{1}{R} P_V + Q_{HA}(t) \right) \quad (9)
\]
\[
\frac{dP_V}{dt} = \frac{1}{C_V} \left( \frac{1}{R} P_A - \frac{1}{R} P_V - Q_{VH}(t) \right) \quad (10)
\]

We can assign numerical values to the problem parameters $R$, $C_A$, and $C_V$ as follows.

\[
R = \frac{P_A - P_V}{Q} \quad \bar{P}_A = 100 \quad \bar{P}_V = 5 \quad \text{and} \quad \bar{Q} = \frac{5000}{60}
\]
\[
C_A = 1.5 \quad C_V = 50.
\]

We will solve this system for various forms of the forcing terms $Q_{HA}$ and $Q_{VH}$.

1. Set up the system of differential equations in the form

\[
\dot{P} = AP + g(t).
\]

Define $P$, $A$, and $g(t)$. Do not use the numerical values for the model parameters.

2. Find the general solution to the corresponding homogeneous system: $\dot{P} = AP$. Again, find this solution in terms of the general parameters.

1. **Constant Forcing:** $Q_{HA} = Q_{VH} = \bar{Q}$

3. Find the general solution of the nonhomogeneous system $\dot{P} = AP + g(t)$.

4. Find the unique solution corresponding to the general initial conditions $P(0) = [P_{A0}, P_{V0}]^{tr}$.

5. Prove that if $P_{A0} - P_{V0} = \bar{P}_A - \bar{P}_V$, then $P(t) = [P_{A0}, P_{V0}]^{tr}$.

   Use Maple to Demonstrate this by solving the system with numerical values assigned to the parameters and initial conditions satisfying the property $P_{A0} - P_{V0} = \bar{P}_A - \bar{P}_V$.

6. If $P_{A0} - P_{V0} \neq \bar{P}_A - \bar{P}_V$ define

\[
\begin{align*}
P^*_A &= \lim_{t \to \infty} P_A(t) \quad \text{and} \quad P^*_V = \lim_{t \to \infty} P_V(t).
\end{align*}
\]

Find $P^*_A$ in terms of $C_A$, $C_V$, $R$, $\bar{Q}$, $P_{A0}$, and $P_{V0}$ and show that $P^*_A - P^*_V = \bar{P}_A - \bar{P}_V$.

Use Maple to Demonstrate this by solving the system with numerical values assigned to the parameters and initial conditions satisfying the property $P_{A0} - P_{V0} \neq \bar{P}_A - \bar{P}_V$.

7. With regards to the above problem, show that $P^*_A$ and $P^*_V$ are independent of $\bar{Q}$ so long as $R = \frac{\bar{P}_A - \bar{P}_V}{\bar{Q}}$.

**Note** When using Maple to solve the system, you should first have Maple solve the system without numerical values assigned to the parameters, and the assign numerical values to the parameters in the solution.
2 Oscillating Forcing

The non-homogeneous terms \( Q_{HA} \) and \( Q_{VH} \) are defined as

\[
Q_{HA}(t) = \overline{Q}(1 + \sin \omega t) \tag{11}
\]
\[
Q_{VH}(t) = \overline{Q}(1 - \sin \omega t) \tag{12}
\]

where

\[
\omega = \frac{2\pi b}{60}, \quad b = 70 \text{ heart rate}
\]

The plots of \( Q_{HA} \) (solid) and \( Q_{VH} \) (dashed) are shown below over one period which lasts just under one second.

8. Have Maple find and plot some solutions for \( P_A(t) \) and \( P_V(t) \) given the prescribed numerical values for the problem parameters, the oscillating forcing terms, and various initial conditions. Notice that there is a transient part and a steady state part to each solution.

9. Express the nonhomogeneous system \( \dot{P} = AP + g(t) \) in the form

\[
\dot{P} = AP + g_1(t) + g_2(t)
\]

where you should have already obtained the particular solution \( (v_1(t)) \) for \( \dot{P} = AP + g_1(t) \). Determine the form of the particular solution \( (v_2(t)) \) for \( \dot{P} = AP + g_2(t) \). Find formulas for the undetermined coefficients involved in \( v_2(t) \). Do not try to calculate these by hand. We’ll have Maple do it later. Determine the general solution to the nonhomogeneous equation \( \dot{P} = AP + g(t) \).

10. Define the unique solution to \( \dot{P} = AP + g(t), \ P_A(0) = P_{AO}, \ P_V(0) = P_{V0} \) in terms of previously defined variables and the initial conditions; \( P_{A0} \) and \( P_{V0} \).

11. In a problem like this, we would like to impose initial conditions so that the final solution is of the form

\[
P_A(t) = \overline{P}_A + a_1 \cos \omega t + b_1 \sin \omega t \tag{13}
\]
\[
P_V(t) = \overline{P}_V + a_2 \cos \omega t + b_2 \sin \omega t \tag{14}
\]

This way the solution is of the form of the mean value plus an oscillation.

Find \( P_{AO} \) and \( P_{V0} \) so that the solution is in this form. Verify that with these initial conditions there is no transient part to the solution.
3 Bonus: Maintaining Arterial Pressure, Nonlinear Models

This is fairly open ended and any relevant insights will get credit.

Let

\[ Q_{HA}(t) = Q_{VH}(t) = f(P_A) \frac{P_A - P_T}{R} \]

Notice that the second term on the right is \( Q_{AV} \) so the heart picks up what is returned to the veins multiplied by \( f(P_A) \). We want \( f(P_A) \) to be such that if \( P_A \) decreases, cardiac output will increase thus increasing \( P_A \) (hopefully) back to its original mean value. We consider two such functions that should achieve this goal.

1. Case (1) \[ f(P_A) = -\frac{P_A}{P_A} + 2 \]

2. Case (2) \[ f(P_A) = \frac{P_A}{P_A} \]

In each case prescribe the initial conditions \( P_A(0) = \overline{P}_A \) and \( P_V(0) = \overline{P}_V \) and try the following scenarios.

- Use \( R = \frac{P_A - P_V}{Q} \)
  
  In this case \( P_A \) and \( P_V \) should remain constant.

- Use \( R = 2\overline{P}_A - \overline{P}_V \)
  
  In this case \( P_A \) should initially increase and \( P_V \) should initially decrease because the resistance between the two compartments has increased. There should be a return to nearly normal pressures after that because cardiac output will decrease.

- Use \( R = \frac{1}{2} \overline{P}_A - \overline{P}_V \)
  
  In this case \( P_A \) should initially decrease and \( P_V \) should initially increase because the resistance between the two compartments has decreased. There should be a return to nearly normal pressures after that because cardiac output will then increase.

Which of the above definitions for \( f(P_A) \) does a better job of returning arterial pressure back to its initial (mean) value of 100 mmHg.