1. Consider the system describing an undamped pendulum

\[ \frac{dx}{dt} = y \quad \frac{dy}{dt} = -\sin x. \]  

(a) Find an equation of the form \( H(x, y) = c \) satisfied by the trajectories.

(b) Plot several level curves of the function \( H \). These are the trajectories of (1). You may find it useful to use Maple and the `implicitplot` command. Be sure to include at least two critical points in this plot and indicate the direction on each trajectory.

2. Prove that if a trajectory starts at a noncritical point of the system

\[ \frac{dx}{dt} = F(x, y) \quad \frac{dy}{dt} = G(x, y) \]

then it cannot reach a critical point \((x_0, y_0)\) in a finite length of time. **Hint** Assume the contrary and use the uniqueness of solutions to arrive at a contradiction.

3. In this problem we show how a small change in the coefficients of a system of linear equations can affect the nature of a critical point when the eigenvalues are equal. Consider the system

\[ \mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x}. \]

Show that the eigenvalues are \( r_1 = -1 \) and \( r_2 = -1 \) so that the critical point \((0, 0)\) is an asymptotically stable (improper) node. Now consider the system

\[ \mathbf{x}' = \begin{pmatrix} -1 & 1 \\ -\epsilon & -1 \end{pmatrix} \mathbf{x} \]

where \(|\epsilon|\) is arbitrarily small. Show that if \( \epsilon > 0 \), then the eigenvalues are \(-1 \pm i\sqrt{\epsilon}\), so that the asymptotically stable node becomes an asymptotically stable spiral point. If \( \epsilon < 0 \), the the roots are \(-1 \pm \sqrt{|\epsilon|}\), and the critical point remains an asymptotically stable node.

4. Consider the system

\[ \frac{dx}{dt} = (2 + x)(y - x) \quad \frac{dy}{dt} = (4 - x)(y + x) \]

(a) Determine all critical points of the system.

(b) Find the corresponding linear system near each critical point.

(c) Find the eigenvalues of each linear system. What conclusions can you draw about the nonlinear system with regards to behavior near the critical points.

(d) Draw a phase portrait of the nonlinear system to confirm your conclusions.