Homework #4 Due: Friday February 27

(50 pts)

MTHBD/CMPBD 424

1. **Stability of the RK2 Method:** Consider the following IVP.

$$x' = \lambda x, \quad x(0) = x_0 \neq 0, \quad \text{and} \quad \lambda < 0$$

Suppose you are to numerically solve this IVP using the RK2 method:

$$x_{k+1} = \frac{h}{2}\left[f(t_k, x_k) + f(t_k + h, x_k + hf(t_k, x_k))\right].$$

Now if $\lambda < 0$ we know the exact solution tends to zero as t tends to infinity. Define the stability constraint on the step size h explicitly in term of λ which ensures that the numerical approximation also tends to zero as t gets large. Describe what will happen to the numerical approximation if this constraint is violated.

2. Cardiac Output may be approximated by the function

$$Q(t) = A\sin^n(\omega t)\ \cos(\omega t - \phi) \tag{1}$$

where $n = 13$, $\phi = \pi/10$, $\omega = 76\pi$ and $A = 106,596$. Figure 1 shows a graph of this function, where the flow is in ml/min and time is in minutes. $p = 1/76$.

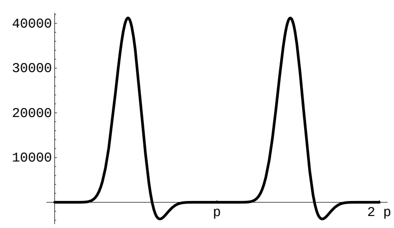


Figure 1: $Q(t)$

A model for Arterial pressure P_A and Venous pressure P_V in mmHg, starts with a few assumptions and applies two conservation equations, (the details are on back), and results in the following system of differential equations for pressures.

$$\frac{dP_A}{dt} = \frac{1}{C_A}\left(-\frac{1}{R}P_A + \frac{1}{R}P_V + Q(t)\right) \quad P_A(0) = \overline{P}_A \tag{2}$$

$$\frac{dP_V}{dt} = \frac{1}{C_V}\left(\frac{1}{R}P_A - \frac{1}{R}P_V - Q(t)\right) \quad P_V(0) = \overline{P}_V \tag{3}$$

where

$$R = \frac{\overline{P}_A - \overline{P}_V}{\overline{Q}} \quad \overline{P}_A = 100 \quad \overline{P}_V = 5 \quad \text{and} \quad \overline{Q} = 6900$$

$$C_A = 2 \quad C_V = 55,$$

and $Q(t)$ is from equation 1.

The above initial value problem can be more easily expressed as

$$d\mathbf{P}/dt = \mathbf{F}(t, \mathbf{P}) \quad \mathbf{P}(0) = \mathbf{P}_o \tag{4}$$

for appropriate choices of $\mathbf{P}$, $\mathbf{P}_o$, and $\mathbf{F}$.

Your Assignment

(a) Write a function file: **Qfun.m** with first line

function Q = Qfun(t)

It should take as input a scalar t representing time. It should output $Q(t)$ from equation (1).

(b) Write a function file: **modelrhs.m** with the first line:

function dPdt = modelrhs(t,P)

which takes as input a scalar t (for time) and a 2x1 column vector of pressures. As output, it should return the 2x1 vector associated with the right hand side of equation 4. You will need to call the function **Qfun** from this program.

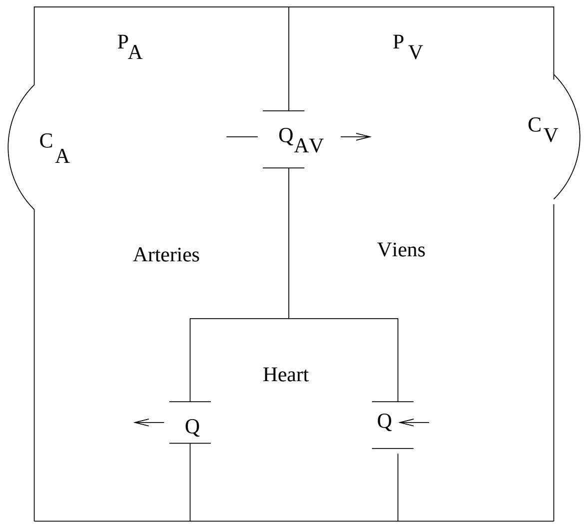
(c) Use the above function file and matlab's canned program **ode45** to solve the system from $t = 0$ to $t = 10/76$. This corresponds to 10 cardiac cycles. In this program apply an upper bound on the approximate relative error of 10^{-6} with the command

[t,Psol] = ode45('modelrhs',10/76,Po,'RelTol',1e-6)

Hand in a plot of P_A over this time period. The numerical solution should settle down to a regular periodic function after the first few periods. What is the maximum arterial pressure and the minimum arterial pressure approximated from this curve after it has settled down? State this on the graph as "*arterial blood pressure is max over min*". According to ER a normal value is about "120 over 80". Also on this graph state the number of time steps that matlab used. You can find this by investigating the length of the solution vector.

(d) Solve the same system using the same **modelrhs.m** function only call it from a program file that solves the system using the Runge -Kutta method of order 2 which we covered in class. You have to program this yourself. Instead of defining a step size, first define the number of steps (n), and then let $h = \frac{10/76}{n}$, and then your vector of t values can be defined as **t = 0:h:10/76**. Plot P_A over the same time interval. Choose the smallest n so that your plot matches that in the above problem. State that value of n on the graph as well as the resulting arterial blood pressure as "max over min". This should be pretty close to the result in the above problem.

A MODEL FOR ARTERIAL AND VENOUS BLOOD PRESSURES



Derivation of the Governing Differential Equations

- Assumption 1:
  The flow between Arteries and Veins Q_{AV} is proportional to the pressure difference.

$$Q_{AV} = \frac{P_A - P_V}{R}$$

  In this equation, R is called the resistance to flow.

- Assumption 2:
  Volume changes are proportional to the pressure changes:

$$\frac{dV_A}{dt} = C_A \frac{dP_A}{dt} \tag{5}$$

$$\frac{dV_V}{dt} = C_V \frac{dP_V}{dt} \tag{6}$$

$$\tag{7}$$

  In these equations, C_A and C_V are the compliances.

- Conservation of mass in each compartment

$$\text{Flow Rate in} - \text{Flow Rate out} = \text{rate of volume change} \tag{8}$$

$$Q(t) - \frac{P_A - P_V}{R} = C_A \frac{dP_A}{dt} \tag{9}$$

$$\frac{P_A - P_V}{R} - Q(t) = C_V \frac{dP_V}{dt} \tag{10}$$

1. Derive the stability constraint on the step size h from problem number 1. You do not have to derive the part I did in class. Describe what will happen to the numerical approximation if this constraint is violated.

Hand In:

- **Page 1:** This page with the answer to problem 1.
- **Page 2:** A copy of **Qfun.m** .
- **Page 3:** A copy of **modelrhs.m**.
- **Page 4:** The graph and comments from problem 2(c).
- **Page 5:** The graph and comments from problem 2(d).
- **Page $\geq$ 6:** The code for problems 2(c) and 2(d).