Consider the heat conduction problem over the interval $x \in [0,1]$.

$$u_{xx} = u_t$$  \hspace{1cm} (1)$$

with initial condition

$$u(x,0) = \sin(\pi x)$$  \hspace{1cm} (2)$$

and the boundary conditions

$$u(0,t) = 0$$  \hspace{1cm} (3)$$

$$u(1,t) = 0$$  \hspace{1cm} (4)$$

We will numerically solve this partial differential equation in various ways and look at the error at the last time step. Let $v$ represent the numerical approximation with space step size $= h$, time step size $= k$, and the number of time steps $= M$.

The exact solution is $u(x, t) = e^{-\pi^2 t} \sin(\pi x)$.

1. Use the \textbf{explicit} method with $h = .1$, $k = 0.005125$, and $M = 200$.

   Plot the relative error $= \frac{v(t) - u(t)}{u(t)}$ at time $t = M \cdot k = 1.025$.

   If $v = u = 0$ set the relative error $= 0$.

   The max $|\text{relative error}|$ at this time is between 0.10 and 0.20.

2. Use the \textbf{explicit} method with $h = .1$, $k = 0.006$, and $M = 171$.

   Plot the relative error $= \frac{v(t) - u(t)}{u(t)}$ at time $t = M \cdot k = 1.026$.

   If $v = u = 0$ set the relative error $= 0$.

   The max $|\text{relative error}|$ at this time is $\approx 10^7$.

3. In class we showed that the explicit method is \textbf{stable} if and only if

   $$\frac{k}{h^2} = s < \frac{1}{(1 - \cos \theta_j)} \quad \text{where} \quad \theta_j = \frac{j\pi}{n + 1} \quad \text{for} \quad 1 \leq j \leq n.$$  

   Here, $n$ is the number of interior $x$-nodes. Show that in problem (1) this constraint is not violated but in problem (2) this constraint is violated.

4. Use the \textbf{implicit} method with $h = .1$, $k = 0.006$, and $M = 171$.

   Plot the relative error $= \frac{v(t) - u(t)}{u(t)}$ at time $t = M \cdot k = 1.026$.

   If $v = u = 0$ set the relative error $= 0$.

   This should resolve the large error from part (2). Why is this?

5. \textbf{Bonus} Turn in a graph of $v(x, t)$ for $x \in [0,1]$, $t \in [0, Mk]$ for problem 1. You’ll have to read about \textbf{mesh} or \textbf{surf} for plotting this.
• In class we showed that the explicit method is **stable** if and only if

\[
\frac{k}{h^2} = s < \frac{1}{(1 - \cos \theta_j)} \quad \text{where} \quad \theta_j = \frac{j\pi}{n+1} \quad \text{for} \quad 1 \leq j \leq n.
\]

Here, \( n \) is the number of interior \( x \)-nodes. Show that in problem (1) this constraint is not violated but in problem (2) this constraint is violated.

Hand In:

• **page 1:** This page with the answer to problem 3.

• **page 2:** The error graph for #1. (put code in my P-drive)

• **page 3:** The error graph for #2. (put code in my P-drive)

• **page 4:** The error graph and explanation for #4. (put code in my P-drive)

• **page 5:** The 3D graph for bonus problem. (put code in my p-drive)