Homework #4 Due: Friday March 3

(50 pts)

MTHBD/CMPBD 424

1. **Stability of the RK2 Method:** Consider the following IVP.

$$x' = \lambda x, \quad x(0) = x_0 \neq 0, \quad \text{and} \quad \lambda < 0$$

Suppose you are to numerically solve this IVP using the RK2 method:

$$x_{k+1} = x_k + \frac{h}{2}\left[f(t_k, x_k) + f(t_k + h, x_k + hf(t_k, x_k))\right].$$

Now if $\lambda < 0$ we know the exact solution tends to zero as t tends to infinity. Define the stability constraint on the step size h explicitly in term of λ which ensures that the numerical approximation also tends to zero as t gets large. Describe what will happen to the numerical approximation if this constraint is violated.

2. Cardiac Output may be approximated by the function

$$Q(t) = A\sin^n(\omega t)\ \cos(\omega t - \phi) \tag{1}$$

where $n = 13$, $\phi = \pi/10$, $\omega = 76\pi$ and $A = 106,596$. Figure 1 shows a graph of this function, where the flow is in ml/min and time is in minutes. $p = 1/76$.

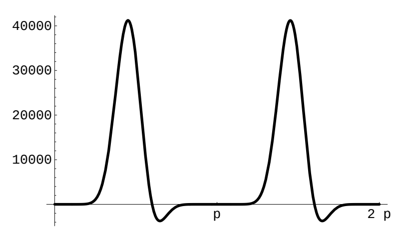


Figure 1: $Q(t)$

A model for Arterial pressure P_A and Venous pressure P_V in mmHg, starts with a few assumptions and applies two conservation equations, (the details are attached), and results in the following system of differential equations for pressures.

$$\frac{dP_A}{dt} = \frac{1}{C_A}\left(-\frac{1}{R}P_A + \frac{1}{R}P_V + Q(t)\right) \quad P_A(0) = \overline{P}_A \tag{2}$$

$$\frac{dP_V}{dt} = \frac{1}{C_V}\left(\frac{1}{R}P_A - \frac{1}{R}P_V - Q(t)\right) \quad P_V(0) = \overline{P}_V \tag{3}$$

where

$$R = \frac{\overline{P}_A - \overline{P}_V}{\overline{Q}} \quad \overline{P}_A = 100 \quad \overline{P}_V = 5 \quad \text{and} \quad \overline{Q} = 6900$$

$$1 \le C_A \le 4 \quad C_V = 55,$$

and $Q(t)$ is from equation 1.

The above initial value problem can be more easily expressed as

$$d\mathbf{P}/dt = \mathbf{F}(t, \mathbf{P}) \quad \mathbf{P}(0) = \mathbf{P}_o \tag{4}$$

for appropriate choices of $\mathbf{P}$, $\mathbf{P}_o$, and $\mathbf{F}$.

Your Assignment for number 2

(a) Write a function file: **Qfun.m** with first line

function Q = Qfun(t)

It should take as input a scalar t representing time. It should output $Q(t)$ from equation (1).

(b) Write a function file: **modelrhs.m** with the first line:

function dPdt = modelrhs(t,P)

which takes as input a scalar t (for time) and a 2x1 column vector of pressures. As output, it should return the 2x1 vector associated with the right hand side of equation 4. You will need to call the function **Qfun** from this program.

(c) Use the above function file and matlab's canned program **ode45** to solve the system from $t = 0$ to $t = 10/76$. This corresponds to 10 cardiac cycles. In this program apply an upper bound on the approximate relative error of 10^{-6} with the command

[t,Psol] = ode45('modelrhs',10/76,Po,odeset('RelTol',1e-6))

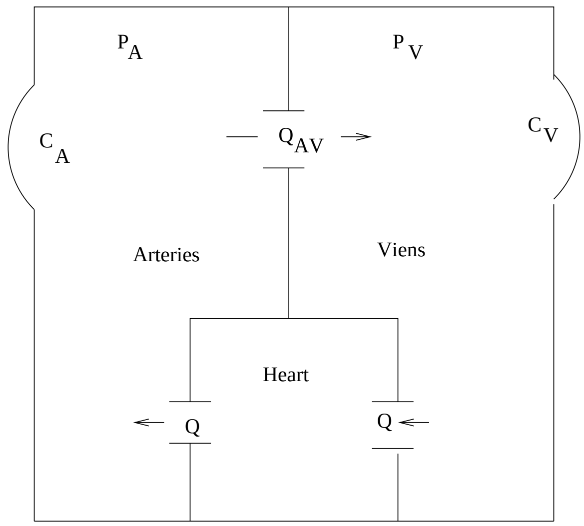
Once you choose a value for C_A you can plot the numerical solution and you should observe that the oscillations 'settle down' to a regular periodic function after several periods. This is the portion of the curve in which we are interested. When $C_A = 1$ you should observe the max pressure over the last few periods should be around 140, and the minimum should be around 70. This person has a blood pressure of "140 over 70" with a pulse amplitude of $140 - 70 = 70$ mmHg. This is bad. Find the value of C_A which results in a "normal" blood pressure of about "120 over 80" and a pulse amplitude that rounds to 40 mmHg. Print a plot the numerical solution using this value of C_A. This properly-labelled plot must include:

- The value of C_A that you found.
- The blood pressure = "max pressure over min pressure".
- The pulse pressure written to two decimal places.
- The number of time steps that ode45 used to get the desired accuracy. You can find this by checking the length of the t-vector.
- Did it use equal sized steps. Again you can find this by investigating the t-vector.

(d) Using the value of C_A which you found above, solve the same system using the same **modelrhs.m** function only call it from a program file that solves the system using the Runge -Kutta method of order 2 which we covered in class. You have to program this yourself. Instead of defining a step size, first define the number of steps (n), and then let $h = \frac{10/76}{n}$, and then your vector of t values can be defined as **t = 0:h:10/76**. Plot P_A over the same time interval. Choose the smallest n so that your the pulse amplitude is between 38 and 42 mmHg. Plot the result using this value of n. This properly-labelled plot must include:

- The number of time steps.
- The blood pressure = "max pressure over min pressure".

A MODEL FOR ARTERIAL AND VENOUS BLOOD PRESSURES



Derivation of the Governing Differential Equations

- Assumption 1:
  The flow between Arteries and Veins Q_{AV} is proportional to the pressure difference.

$$Q_{AV} = \frac{P_A - P_V}{R}$$

  In this equation, R is called the resistance to flow.
- Assumption 2:
  Volume changes are proportional to the pressure changes:

$$\begin{aligned} \frac{dV_A}{dt} &= C_A \frac{dP_A}{dt} && (5) \\ \frac{dV_V}{dt} &= C_V \frac{dP_V}{dt} && (6) \\ & && (7) \end{aligned}$$

  In these equations, C_A and C_V are the compliances.
- Conservation of mass in each compartment

$$\begin{aligned} \text{Flow Rate in} - \text{Flow Rate out} &= \text{rate of volume change} && (8) \\ Q(t) - \frac{P_A - P_V}{R} &= C_A \frac{dP_A}{dt} && (9) \\ \frac{P_A - P_V}{R} - Q(t) &= C_V \frac{dP_V}{dt} && (10) \end{aligned}$$

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1. Derive the stability constraint on the step size h from problem number 1. You do not have to derive the part I did in class. Describe what will happen to the numerical approximation if this constraint is violated.

Hand In:

- **Page 1:** This page with the answer to problem 1.
- **Page 2:** The graph and data from problem 2(c).
- **Page 3:** The graph and data from problem 2(d).
- The code for problems 2(a) – 2(d) should be placed in my P-drive.