Numerical Integration (intro)
MTHBD/CMPBD 423

• Question: What is \( \int_a^b f(x) \, dx \)?

Suppose the interval \([a, b]\) is divided into a partition \(a = x_0 < x_1 < x_2 < \ldots < x_n = b\), where \(\Delta x_i = x_{i+1} - x_i\). Furthermore, assume that as \(n \to \infty\), \(\Delta x_i \to 0\) for all \(i\) and that \(x^*_i\) is some point in the \(i\)’th subinterval. Then

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x^*_i) \Delta x_i
\]

provided this limit exists. If this limit exists then the function is called integrable on \([a, b]\).

• Theorems on integrable functions:

If \(f\) is continuous on \([a, b]\), then \(f\) is integrable on \([a, b]\). Furthermore if \(f\) is piecewise continuous on \([a, b]\) then \(f\) is integrable on \([a, b]\).

If \(f\) is continuous on \([a, b]\) and \(F\) is any antiderivative of \(f\) on \([a, b]\) then

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

Problem: Many times we cannot find an antiderivative but we would like to approximate the value of a given definite integral.

Solution: Approximate the function with another function that we know how to integrate (usually a polynomial). If the approximating function does a good job then integrating this should result in a good approximation of the desired definite integral.

• Definition

Suppose \(a = x_0 < x_1 < \ldots < x_M = b\). A formula of the form

\[
Q[f] = \sum_{k=0}^M \omega_k f(x_k) = \omega_0 f(x_0) + \ldots + \omega_M f(x_M)
\]

with the property that

\[
\int_a^b f(x) \, dx = Q[f] + E[f]
\]

is called a numerical integration or quadrature formula. The term \(E[f]\) is called the truncation error. The values \(\{x_k\}_{k=0}^M\) are called the quadrature nodes, and \(\{\omega_k\}_{k=0}^M\) are called the weights.

• Definition

The order of precision of a quadrature formula is the maximum \(n\) for which \(E[P_k] = 0\) for all \(k \leq n\).

It should not be surprising then that the truncation error for a quadrature formula with a degree of precision \(n\) is of the form

\[
E[f] = k f^{(n+1)}(\xi)
\]

This way, if \(f = P_n\), \(E[f] = 0\).

• Definition

If the quadrature formula is derived by interpolating \(M + 1\) evenly space points \(\{(x_k, f_k)\}_{k=0}^M\) with \(P_M(x)\) and then integrating \(P_M(x)\), the formula is called a Newton-Cotes formula.

Most of the quadrature formulas you currently know are Newton-Cotes formulas.
**Definition**

If $x_0 = a$ and $x_M = b$ the associated Newton-Cotes formula is called a **closed** Newton-Cotes formula.

A **composite** formula is a simplified sum of closed formulas. Most formulas in Calculus books are composite formulas.

**Closed Newton-Cotes Quadrature Formulas** $x_k = x_0 + kh$ and $f_k = f(x_k)$.

- **Trapezoid Rule:**
  \[
  \int_a^b f(x) \, dx \approx \frac{h}{2} (f_0 + f_1) \quad \text{Local Error} = -\frac{1}{12} h^3 f''(\xi)
  \]

- **Simpson’s $\frac{1}{3}$ Rule:**
  \[
  \int_a^b f(x) \, dx \approx \frac{h}{3} (f_0 + 4f_1 + f_2) \quad \text{Local Error} = -\frac{1}{90} h^5 f^{(4)}(\xi)
  \]

- **Simpson’s $\frac{3}{8}$ Rule:**
  \[
  \int_a^b f(x) \, dx \approx \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3) \quad \text{Local Error} = -\frac{3}{80} h^5 f^{(4)}(\xi)
  \]

- **Boole’s Rule:**
  \[
  \int_a^b f(x) \, dx \approx \frac{2h}{45} (7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) \quad \text{Local Error} = O(h^6) f^{(5)}(\xi)
  \]

**Orders of precision:**

- Trapezoid Rule: 1
- Simpson’s $1/3$ rule: 3
- Simpson’s $3/8$ rule: 3
- Boole’s Rule: 4

Notice the bonus precision and local truncation error with Simpson’s $1/3$ rule.