Numerical Differentiation (intro)
MTHBD/CMPBD 423

• **Difference Quotients**: \( \frac{f(x) - f(a)}{x - a} \).

The derivative is by definition the limit of difference quotients:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

provided this limit exists.

So it is safe to assume that

\[
f'(x) \approx \frac{f(x + h) - f(x)}{h}
\]

for small \( h \).

The above formula is called a **forward difference formula**. We have two main questions to ask about such difference formulas.

1. How small shall we make \( h \)?
2. How accurate is the approximation?

Bad answer to Question (1): **The smaller the better**

Roundoff error and loss of significance can occur in the numerator of equation 1.

Example: Let \( h_k = 10^{-k} \), \( f(x) = e^x \) and use equation 1 to estimate \( f'(1) = e \). Let

\[
D_k = \frac{f(1 + h_k) - f(1)}{h_k}
\]

\[
\text{error} = E_k = |D_k - e|
\]

The graph below shows that the error decreases as \( h \) decreases from \( 10^{-2} \) to \( 10^{-8} \) but then increases when \( h \) is smaller than \( 10^{-8} \). Moral: **smaller \( h \) is not necessarily better**! This graph was generated using MATLAB working at full precision, this is not a contrived machine with small precision.

Goal: Develop formulas that give good accuracy for large \( h \) values.