**Divided Differences and Newton’s Interpolation Polynomial**

- **The Problem**
  Interpolate a function $f$ at $n+1$ distinct values of $x$ using the Newton Interpolation Polynomial by calculating the coefficients of this polynomial using the divided differences of $f$.
  
  The *divided difference polynomial* is just Newton’s interpolating polynomial applied to this type of problem.

- **Newton Interpolation Polynomial**
  
  $$p_n(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) + \ldots + c_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

- **Divided Difference Polynomial**
  
  We define the first two divided differences as follows
  
  $$f[x_0] = c_0 = f(x_0)$$
  
  $$f[x_0,x_1] = c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$
  
  These first two are simple and you can see why the phrase *divided differences* is used. We will define the remaining divided differences by
  
  $$f[x_0,x_1,x_2,\ldots,x_n] = c_n$$
  
  and the divided difference polynomial becomes
  
  $$p_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \ldots + f[x_0,x_1,\ldots,x_n](x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

- **The theorem that makes calculating these divided differences possible**
  
  Divided differences satisfy
  
  $$f[x_0,x_1,x_2,\ldots,x_n] = \frac{f[x_1,x_2,\ldots,x_n] - f[x_0,x_1,\ldots,x_{n-1}]}{x_n - x_0}$$
  
  **Proof**: Let $p_n$ denote the polynomial that interpolates $f$ at $x_0,x_1,\ldots,x_n$. Let $q$ denote the polynomial that interpolates $f$ at $x_1,x_2,\ldots,x_n$. I claim
  
  $$p_n(x) = q(x) + \frac{x-x_n}{x_n-x_0}[q(x) - p_{n-1}(x)].$$
  
  Proof of claim: both sides are polynomials of degree $\leq n$ and both sides evaluate to $f(x_i)$ for $0 \leq i \leq n$. Therefore the polynomials must be the same (claim proved). Equating the coefficient of the $x^n$ on both sides yields equation (3) $\Diamond$.

  Now we can now expand this to arbitrary starting values of $x_i$:
  
  $$f[x_i,x_{i+1},\ldots,x_{i+j}] = \frac{f[x_{i+1},x_{i+2},\ldots,x_{i+j}] - f[x_i,x_{i+2},\ldots,x_{i+j-1}]}{x_{i+j} - x_i}$$
• Defining divided differences.

\[
\begin{align*}
\frac{f[x_0]}{} & = f(x_0) \\
\frac{f[x_0, x_1]}{} & = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_0) - f(x_0)}{x_1 - x_0}
\end{align*}
\]

The divided differences table:

• Defining divided differences.

Here’s an algorithm for creating the above table given the vector \( c \) given by the first row in the above table.

\[
\text{The coefficients in the divided difference polynomial and hence the Newton Interpolation Polynomial are given by the first row in the above table.}
\]

• In Simpler form:

\[
\begin{align*}
\begin{array}{cccccc}
x_0 & f(x_0) & f[x_0, x_1] & f[x_0, x_1, x_2] & f[x_0, x_1, x_2, x_3] & f[x_0, x_1, x_2, x_3, x_4] \\
x_1 & f(x_1) & f[x_1, x_2] & f[x_1, x_2, x_3] & f[x_1, x_2, x_3, x_4] \\
x_2 & f(x_2) & f[x_2, x_3] & f[x_2, x_3, x_4] \\
x_3 & f(x_3) & f[x_3, x_4] \\
x_4 & f(x_4) \\
\end{array}
\end{align*}
\]

You are given the data in the first two columns and have to determine the remaining columns in order by the algorithm defined by equation (4) repeated below:

\[
f[x_i, x_{i+1}, \ldots, x_{i+j}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+j}] - f[x_i, x_{i+2}, \ldots, x_{i+j-1}]}{x_{i+j} - x_i}
\]

The coefficients in the divided difference polynomial and hence the Newton Interpolation Polynomial are given by the first row in the above table.

Here’s an algorithm for creating the above table given the vector \( x \) and the first column of \( c \).

\[
\begin{align*}
\text{for } j = 1:4 \\
\text{for } i = 0:n-j \\
c_{i,j} &= (c_{i+1,j-1} - c_{i,j-1})/(x_{i+j} - x_i) \\
\text{end (i loop)} \\
\text{end (j loop)}
\end{align*}
\]

An Example for five data points \( (x_i, f(x_i)) \quad 0 \leq i \leq 4 \):
• **Theorem on Error in Newton Interpolation**

Let $p$ be the polynomial of degree $\leq n$ which interpolates $f$ at a set of $n + 1$ distinct nodes, $x_0, x_1, \ldots, x_n$. If $t$ is a point different from one of the nodes then

$$f(t) - p(t) = f[x_0, x_1, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j)$$

**Proof:** Let $q$ denote the polynomial that interpolates $f$ at $x_0, x_1, \ldots, x_n, t$. Then $q$ is obtained from $p$ by

$$q(x) = p(x) + f[x_0, x_1, \ldots, x_n, t] \prod_{j=0}^{n} (x - x_j)$$

Since $q(t) = f(t)$ we obtain the desired result ◇.

• **Theorem on derivatives and divided differences**

If $f \in C^n[a, b]$ and if $x_0, x_1, \ldots, x_n$ are distinct points in $[a, b]$ then there is an $\eta$ in $(a, b)$ such that

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^n(\eta)$$

**Proof:** Let $p$ be the polynomial of degree $\leq n - 1$ that interpolates $f$ at the nodes $x_0, x_1, \ldots, x_{n-1}$. By the approximation error theorem there exists an $\eta$ in $(a, b)$ such that

$$f(x_n) - p(x_n) = \frac{1}{n!} f^n(\eta) \prod_{j=0}^{n-1} (x_n - x_j).$$

By the above theorem this can be expressed as

$$f(x_n) - p(x_n) = f[x_0, x_1, \ldots, x_n] \prod_{j=0}^{n-1} (x_n - x_j)$$

Setting these equal to each other yields the desired result ◇.

This theorem will help us out later when trying to approximate $n$th order derivatives.

• **Weierstrass Approximation Theorem**

If $f$ is continuous on $[a, b]$ and if $\varepsilon > 0$ is given, then there is a polynomial $p$ satisfying $|f(x) - p(x)| \leq \varepsilon$ on the interval $[a, b]$.

The proof is tricky and requires the use of **Bernstein Polynomials**. We will not go into this except to say: *Interpolating polynomials will not necessarily converge to a given function.*
FYI material not on exam or homework ... maybe not even covered

New notation:

\[ f_i^{[n]} = f[x_i, x_{i+1}, \ldots x_{i+n}] \]

is the \( n \)th order divided difference at \( x_i \).

- **Newton-Gregory Forward Difference Polynomial** (evenly spaced data): \( x_i \)'s represent \( N+1 \) evenly spaced \( x \)-values where \( x_{i+1} - x_i = h \) and \( s = \frac{x-x_0}{h} \).

\[
P_n(x) = f_0 + s \Delta f_0 + \frac{s(s-1)}{2!} \Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!} \Delta^3 f_0 + \ldots + \frac{s(s-1)\ldots(s-n+1)}{n!} \Delta^n f_0
\]

where

\[
\begin{align*}
    f_i &= f(x_i) \quad \text{for } i = 0 \ldots N \\
    \Delta f_i &= f_{i+1} - f_i \quad \text{for } i = 0 \ldots N-1 \\
    \Delta^2 f_i &= f_{i+2} - 2f_{i+1} + f_i \quad \text{for } i = 0 \ldots N-1 \\
    \vdots \\
    \Delta^n f_i &= f_{i+n} - nf_{i+n-1} + \frac{n(n-1)}{2!}f_{i+n-2} - \ldots \pm f_i, \quad \text{for } i = 0 \ldots N-n
\end{align*}
\]

- Comparing the coefficients of the Divided Difference Polynomial and the Newton-Gregory Forward Difference Polynomial. (only valid if the data are evenly spaced in both cases)

\[
f_i^{[n]} = \frac{\Delta^n f_i}{n! h^n}
\]

- **Error**: The error associated with approximating \( f(x) \) with \( P_n(x) \) is then given by

\[
f(x) = P_n(x) + E_n(x) \tag{8}
\]

\[
E_n(x) = (x-x_0)(x-x_1) \ldots (x-x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!} \tag{9}
\]

where \( \xi \) is in the smallest interval containing \( \{x, x_0, x_1, \ldots x_n\} \) and depends on \( x \).

When \( f(x) \) is not known you can use *The Next Term Rule*.

\[
E_n(x) \approx \text{The value of the next term added to } P_n(x)
\]

or

\[
E_n(x) \approx P_{n+1}(x) - P_n(x)
\]