Bisection Method
MTHBD 423

1. Based on the Intermediate Value Theorem (continuous functions).

2. The Bisection Method of Bolzano (Interval Halving)

   - Start with an initial interval \( [a, b] \) where \( f(a) \) and \( f(b) \) have opposite signs. (graph or search)
     By the Intermediate value theorem, there exists \( r \in (a, b) \) such that \( f(r) = 0. \)

   - Let \( c = \frac{b + a}{2} \)

     - If \( f(a) \) and \( f(c) \) have opposite signs: zero is in \( [a, c] \).
     - If \( f(c) \) and \( f(b) \) have opposite signs: zero is in \( [b, c] \).
     - If \( f(c) = 0 \), the a zero occurs at \( c \).

   In either of the first two cases, the new interval is one half the width of the original. Label this new interval \( [a, b] \) and do it again.

   - first interval is \( [a_0, b_0] \) and \( c_0 = (a_0 + b_0)/2 \)
   - second interval is \( [a_1, b_1] \) and \( c_1 = (a_1 + b_1)/2 \)
     where \( a_1 = c_0 \) and \( b_1 = b_0 \) or \( a_1 = a_0 \) and \( b_1 = c_0 \)
   - \( n \)’th interval is \( [a_n, b_n] \) and \( c_n = (a_n + b_n)/2 \)
     where \( a_n = c_{n-1} \) and \( b_n = b_{n-1} \) or \( a_n = a_{n-1} \) and \( b_n = c_{n-1} \)
   - \( \{a_n\}_{n=0}^{\infty} \) is an increasing sequence.
   - \( \{b_n\}_{n=0}^{\infty} \) is an decreasing sequence.
   - and \( a_n \leq r \leq b_n \) for all \( n \).

   - **Theorem** (Bisection Theorem) Assume \( f \in C[a, b] \) and that \( f(a) \) and \( f(b) \) are nonzero of opposite sign. Then there exists a number \( r \in (a, b) \) such that \( f(r) = 0 \) and the sequence of \( c_n \)’s generated by the bisection process satisfies

     \[
     \lim_{n \to \infty} c_n = r.
     \]

**Proof**

(a) Existence: Consider \( g(x) = f(x) - x \), Notice \( g(a) \cdot g(b) < 0 \), Therfore by IVT \( r \in (a, b) \) exists such that \( f(r) = 0. \)

(b) Convergence

   - \( |e_0| = |c_0 - r| \leq \frac{b_0 - a_0}{2} \)
   - \( |e_1| = |c_1 - r| \leq \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2} \)
   - ... by induction
   - \( |e_n| = |c_n - r| \leq \frac{b_n - a_n}{2^n} \)

   - by the squeezing theorem on \( 0 \leq |c_n - r| \leq \frac{b_0 - a_0}{2^n} \)

   - \( \lim_{n \to \infty} |c_n - r| = 0. \)

*note: This doesn’t really show that \( f(\lim c_n) = 0. \) A more complete proof is given in class.

- Number of iterations -vs- error: Suppose you want to approximate \( r \) with an error \( \leq \varepsilon \), How many iterations should you perform in terms of \( \varepsilon \) and the original interval length. Homework problem.