• Bezier Curves: (P. Bezier of Renault Automobile Company)

1. Bezier curves only interpolate the first and last data points.
2. The points \( P_i = (x_i, y_i) \) are not generally data points but are called control points.
3. The curve stays within the polygon (convex hull) determined by the control points.
4. Changing one control point has a more localized effect on the shape of the curve than with cubic splines or polynomial interpolation.
5. **Construction.** Given \( n + 1 \) control points, let

\[
P_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \quad \text{and} \quad P(u) = \begin{bmatrix} x(u) \\ y(u) \end{bmatrix} \quad \text{for} \quad u \in [0, 1].
\]

Then

\[
P(u) = \sum_{i=0}^{n} \binom{n}{i} (1-u)^{n-i} u^i P_i \quad \text{for} \quad 0 \leq u \leq 1.
\]

Here, the binomial coefficient is used

\[
\binom{n}{i} = \text{“n choose i”} = \frac{n!}{i! (n-i)!}.
\]

Notice

\( P(0) = P_0 \) and \( P(1) = P_n \)

and the first and last control points are interpolated. But these only.

6. **Example** with three points.

\[
P(u) = 1 (1-u)^2 P_0 + 2 (1-u) u P_1 + 1 u^2 P_2
\]

\[
\begin{bmatrix} x(u) \\ y(u) \end{bmatrix} = (1-u)^2 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + (1-u) u \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + u \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}
\]

7. Convex Hull Property:

The Bezier curve is contained in the convex hull determined by the control points. The convex hull of a set of points is the smallest convex set that contains these points. A convex set is one where the line connecting any two points in the set is contained entirely in the set.

8. Initial and Terminal Slope The initial slope heads in the direction of the second point. The final slope points directly away from the second to last point.

9. **Demonstration** Run bezann.m
Beziers Curves and B-Splines

- B-splines

1. Part Bezier curve part Spline. We will be dealing with cubic B-splines.

2. The data points are considered control points. They are not interpolated. It can be manipulated to run through any particular points but that is generally not the case.

3. **Construction** Given the points \( P_i = (x_i, y_i) \) for \( i = 0, 1, \ldots, n \), the cubic B-spline for the interval \((P_i, P_{i+1})\) for \( i = 1, 2, \ldots, n-1 \) is

\[
B_i(u) = \sum_{k=-1}^{2} b_k(u)P_{i+k}
\]

where

\[
\begin{align*}
  b_{-1} &= \frac{(1-u)^3}{6} \\
  b_0 &= \frac{u^3}{2} - u^2 + \frac{2}{3} \\
  b_1 &= -\frac{u^3}{2} + \frac{u^2}{2} + \frac{u}{2} + \frac{1}{6} \\
  b_2 &= \frac{u^3}{6} \quad \text{for } 0 \leq u \leq 1
\end{align*}
\]

Notice: It takes 4 points to generate one portion of the curve. For example, \( B_1(u) \) requires \( P_0, P_1, P_2, \) and \( P_3 \). It does not interpolate any of these.

4. Like a cubic spline, cubic B-splines maintain second derivative continuity.

5. Like Bezier curves, each portion of the curve remains in the convex hull of the four points determining that portion.

6. If you want the curve to go through \( P_0 \) for example you introduce two more points at the beginning, specifically \( P_{-1} = P_0 \) and \( P_{-2} = P_0 \). A similar process can ensure that the curve goes through the last point.

7. Periodic curves can be achieved by introducing extra points as well. If \( P_0 = P_n \), then \( P_{-1} = P_{n-1} \), \( P_{-2} = P_{n-2} \), and \( P_{n+1} = P_1, P_{n+2} = P_2 \).

8. **Demonstration:** Run Bsplineann.m