1. Find the singular point of the given differential equation and classify it as regular or irregular. Justify your classification. (10 pts)

\[ x^2 y'' + 2y' + 3y = 0 \]

The only singular point is at \( x_0 = 0 \). Notice \( \lim_{x \to 0} \frac{2}{x^2} = \lim_{x \to 0} \frac{2}{x} \) is not finite and therefore the singular point is \textbf{irregular}. There is no need to check the second condition for a regular singular point.

2. Solve the following initial value problem. Assume \( x > 0 \). (10 pts)

\[ x^2 y'' - 5xy' + 9y = 0 \quad y(1) = 3 \quad \text{and} \quad y'(1) = 2 \]

Assume a solution of the form \( y = x^r \). Finding \( y' \) and \( y'' \) leads to \( x^r [r(r - 1) - 5r + 9] = 0 \rightarrow (r - 3)^2 = 0 \) so \( r = 3 \) is a double root. Therefore the general solution is \( y(x) = c_1 x^3 + c_2 x^3 \ln(x) \). Now \( y'(x) = 3c_1 x^2 + 3c_2 x^2 \ln(x) + c_2 x^{3 \frac{1}{2}} \). So \( y(1) = c_1 \) and \( y'(1) = 3c_1 + c_2 \). Imposing the initial conditions gives \( c_1 = 3 \) and \( c_2 = -7 \). So \( y(x) = 3x^3 - 7x^3 \ln(x) \).

3. Find the inverse Laplace transform of the following functions. (20 pts)

(a) \[ F(s) = \frac{2e^{-2s}}{s^2 - 4} = e^{-2s} \frac{2}{s^2 - 4} = e^{-2s} \frac{2}{s^2 - 2^2} \]

\[ f(t) = u_2(t)g(t - 2) \text{ where } g(t) = \mathcal{L}^{-1} \left\{ \frac{2}{s^2 - 2^2} \right\} = \sinh 2t \text{ Therefore, } f(t) = u_2(t) \sinh 2(t - 2) \]

(b) \[ F(s) = \frac{s + 4}{(s - 1)^2 + 4} = \frac{(s - 1) + 5}{(s - 1)^2 + 4} = \frac{s - 1}{(s - 1)^2 + 4} + \frac{5}{2} \frac{2}{(s - 1)^2 + 4} \]

Therefore \( f(t) = e^t \cos (2t) + \frac{5}{2} \sin (2t) \).

4. Find the Laplace transform of the following function. (10 pts)

\[ f(t) = \begin{cases} 
0 & t < 2 \\
(t - 2)^3 & t \geq 2 
\end{cases} \]

\[ f(t) = u_2(t)g(t - 2) \text{ where } g(t) = t^3 \]. Therefore \( F(s) = e^{-2s}G(s) \) where \( G(s) = \mathcal{L} \{t^3\} = \frac{3!}{s^4} \).
5. Use the Laplace transform to solve the initial value problem \((20 \text{ pts})\)

\[y'' + 2y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 2\]

Taking the Laplace transform of the entire equation yields

\[s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] - 3Y(s) = 0\]

plugging in the initial conditions

\[s^2Y(s) - s - 2 + 2sY(s) - 2 - 3Y(s) = 0\]

\[Y(s)[s^2 + 2s - 3] = s + 4\]

\[Y(s) = \frac{s+4}{s^2 + 2s - 3} = \frac{s+4}{(s+3)(s-1)}\]

partial fractions gives \(Y(s) = \frac{-1}{4} \left( \frac{1}{s+3} \right) + \frac{5}{4} \left( \frac{1}{s-1} \right)\).

Taking the inverse Laplace transform yields

\[y(t) = \frac{-1}{4} e^{-3t} + \frac{5}{4} e^t.\]

6. Find the solution of the initial value problem \((30 \text{ pts})\)

\[y'' + 9y = u_2(t), \quad y(0) = 1, \quad y'(0) = 2\]

Taking the Laplace transform of the entire equation yields

\[s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{e^{-2s}}{s}\]

plugging in the initial conditions

\[s^2Y(s) - s - 2 + 9Y(s) = \frac{e^{-2s}}{s}\]

\[Y(s)[s^2 + 9] = 2 + s + \frac{e^{-2s}}{s}\]

\[Y(s) = \frac{2}{s^2 + 9} + \frac{s}{s^2 + 9} + e^{-2s} \left( \frac{1}{s^2 + 9} \right)\]

Partial fractions on the second factor of the last term yields \(\frac{1}{s(s^2 + 9)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 9}\) where \(A = 1/9\), \(B = -1/9\), and \(C = 0\).

So

\[Y(s) = \frac{2}{3} \left( \frac{3}{s^2 + 9} \right) + \frac{s}{s^2 + 9} + e^{-2s} \left( \frac{1}{s} - \frac{s}{s^2 + 9} \right)\]

and taking the inverse Laplace transform yields

\[y(t) = \frac{2}{3} \sin (3t) + \cos (3t) + \frac{u_2(t)}{9} h(t-2) \text{ where } h(t) = \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{s}{s^2 + 9} \right] = 1 - \cos (3t)\]

So

\[y(t) = \frac{2}{3} \sin (3t) + \cos (3t) + \frac{u_2(t)}{9} [1 - \cos (3(t-2))]\]