Derivation of Fourier Coefficients

Suppose a Fourier series converges to a function $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m \pi x}{L} + b_m \sin \frac{m \pi x}{L} \right)$$  \quad (1)$$

The goal is to figure out the coefficients $a_m$ and $b_m$.

- The functions $\sin \left( \frac{m \pi x}{L} \right)$ and $\cos \left( \frac{m \pi x}{L} \right)$ for $m = 1, 2, \ldots$ satisfy the following properties.

$$\int_{-L}^{L} \cos \frac{m \pi x}{L} \cos \frac{n \pi x}{L} \; dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$  \quad (2)$$

$$\int_{-L}^{L} \cos \frac{m \pi x}{L} \sin \frac{n \pi x}{L} \; dx = 0 \quad \text{for all } m \text{ and } n$$  \quad (3)$$

$$\int_{-L}^{L} \sin \frac{m \pi x}{L} \sin \frac{n \pi x}{L} \; dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$  \quad (4)$$

- The Euler-Fourier Formulas: Multiply equation 1 by $\sin \left( \frac{m \pi x}{L} \right)$ or $\cos \left( \frac{m \pi x}{L} \right)$ and integrate from $-L$ to $L$ to obtain a formula for each coefficient.

For example; multiplying equation 1 by $\cos \left( \frac{n \pi x}{L} \right)$ and integrating from $-L$ to $L$ yields;

$$\int_{-L}^{L} f(x) \cos \frac{n \pi x}{L} \; dx = \frac{a_0}{2} \int_{-L}^{L} \cos \frac{n \pi x}{L} \; dx + \sum_{m=1}^{\infty} \left( a_m \int_{-L}^{L} \cos \frac{m \pi x}{L} \cos \frac{n \pi x}{L} \; dx + b_m \int_{-L}^{L} \sin \frac{m \pi x}{L} \cos \frac{n \pi x}{L} \; dx \right)$$  \quad (5)$$

Notice: The first integral after the equal sign is zero (assuming $n \neq 0$). All of the integrals after the $b_m$’s are zero, and all of the integrals after the $a_m$’s are zero unless $m = n$ in which case the integral is $L$. So we are left with

$$\int_{-L}^{L} f(x) \cos \frac{n \pi x}{L} \; dx = a_n \; L \quad \text{for } n = 1, 2, \ldots$$  \quad (6)$$

Integrating equation 1 from $-L$ to $L$ shows that this formula also works for $n = 0$. Solving the above equation for $a_n$ gives

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n \pi x}{L} \; dx \quad \text{for } n = 0, 1, 2, \ldots$$  \quad (7)$$

Playing the same game only multiplying equation 1 by $\sin \left( \frac{n \pi x}{L} \right)$ and integrating from $-L$ to $L$ yields;

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n \pi x}{L} \; dx \quad \text{for } n = 1, 2, \ldots$$  \quad (8)$$

Equations (7) and (8) are known as the Euler-Fourier formulas for the coefficients in a Fourier series.