Power Series Method and Taylor Series Method

Consider an initial value problem of the form

\[ y'' + p(x)y' + q(x)y = g(x), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \]

• **Power Series Method:** Assume a power series solution of the form

\[ y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \ldots \]

Notice, \( y(x_0) = a_0 \), so \( a_0 = y_0 \), and \( y'(x_0) = a_1 \), so \( a_1 = y'_0 \), and we already have the first two terms.

Now we plug this form of the solution into the differential equation where derivatives of \( y \) are taken term by term. By equating like powers of \((x - x_0)\) we can derive a recurrence relation between \( a_n \) and previous \( a \)'s. Since we have \( a_0 \) and \( a_1 \), we work from here to find \( a_2 \), then \( a_3 \) and so on until we have the desired number of terms.

If \( p, q, \) or \( g \), are not polynomials in \((x - x_0)\) we must express them as such and this may entail expanding transcendental functions (sine, cosine, exponential, logarithmic, etc.) as a Taylor Series about \( x_0 \).

• **Taylor Series Method:** Assume a Taylor series solution of the form

\[ y = \sum_{n=0}^{\infty} y^{(n)}(x_0) \frac{(x - x_0)^n}{n!} = y(x_0) + y'(x_0)(x - x_0) + y''(x_0) \frac{(x - x_0)^2}{2!} + y'''(x_0) \frac{(x - x_0)^3}{3!} + \ldots \]

where \( y^{(n)}(x_0) \) represents the \( n \)'th derivative of \( y \) evaluated at \( x_0 \).

Notice similarly that the first two terms are given by the initial conditions. To find the third term we need \( y''(x_0) \). We go back to the differential equation to find this. Rewriting the differential equation as

\[ y'' = -p(x)y' - q(x)y + g(x) \]

allows us to plug in \( x = x_0 \) and find \( y''(x_0) \). Now to find \( y'''(x_0) \) we differentiate the above equation and get

\[ y''' = -p'y' - py'' - q'y - qy' + g' \]

So we can plug \( x_0 \) into the right hand side to get \( y'''(x_0) \). Keep going until you have enough terms (whatever that means). Warning: This can get messy but it is possible.