Nonhomogeneous Equations

A nonhomogeneous equation of the form

\[ ay'' + by' + cy = g(t) \]

has a general solution of the form

\[ y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \]

where \( y_1(t) \) and \( y_2(t) \) are linearly independent solutions to the corresponding homogeneous equation and \( Y(t) \) is a particular solution to the nonhomogeneous equation. This handout deals with finding \( Y(t) \).

Method of Undetermined Coefficients

This method can only be used when the nonhomogeneous term \( g(t) \) contains polynomial, exponential, sine or cosine functions exclusively.

\[ ay'' + by' + cy = g_1(t) + g_2(t) + \cdots + g_m(t) \]

- If \( g_i(t) = P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n, \)
  let \( Y_1(t) = t^s \left( A_0 t^n + A_1 t^{n-1} + \cdots + A_n \right), \)
  where \( s \) is the number of times \( 0 \) is a root of the characteristic equation.
  Then substitute \( Y_1(t) \) into \( ay'' + by' + cy = g_i \) and solve for the unknown coefficients: \( A_0, \ldots, A_n. \)
- If \( g_i(t) = P_n(t) e^{\alpha t}, \)
  let \( Y_1(t) = t^s \left( A_0 t^n + A_1 t^{n-1} + \cdots + A_n \right) e^{\alpha t}, \)
  where \( s \) is the number of times \( \alpha \) is a root of the characteristic equation.
  Then substitute \( Y_1(t) \) into \( ay'' + by' + cy = g_i \) and solve for the unknown coefficients: \( A_0, \ldots, A_n. \)
- If \( g_i(t) = P_n(t) e^{\alpha t} \cos \beta t \)
  let \( Y_1(t) = t^s \left( A_0 t^n + A_1 t^{n-1} + \cdots + A_n \right) e^{\alpha t} \cos \beta t + \left( B_0 t^n + B_1 t^{n-1} + \cdots + B_n \right) e^{\alpha t} \sin \beta t, \)
  where \( s \) is the number of times \( \alpha + i\beta \) is a root of the characteristic equation.
  Then substitute \( Y_1(t) \) into \( ay'' + by' + cy = g_i \) and solve for the unknown coefficients: \( A_0, \ldots, A_n, \)
  and \( B_0, \ldots, B_n. \)

Notes: If the polynomial \( P_n(t) \) is a constant, then each polynomial in the assumed form of \( Y_1 \) is a constant as well. \( s \) is the smallest positive integer which ensures no term in the sum describing the particular solution is a solution of the corresponding homogeneous equation.

Finally, the particular solution is given by

\[ Y(t) = Y_1(t) + Y_2(t) + \cdots + Y_m(t). \]

Variation of Parameters

This method may be used for nonhomogeneous terms of any form. The particular solution is given by

\[ Y(t) = -y_1(t) \int \frac{y_2(t) \ g(t)}{W(y_1, y_2)(t)} \ dt + y_2(t) \int \frac{y_1(t) \ g(t)}{W(y_1, y_2)(t)} \ dt. \]