1. Definitions

(a) The $n \times n$ matrix with ones along the diagonal and zeros elsewhere is called the **identity matrix** and is denoted $I_n$.

(b) If $A$ is a matrix $(a_{ij})$, the **transpose** of $A$, denoted $A^T$, is defined by $a_{ij}^T = a_{ji}$.

(c) If $A^T = A$, then $A$ is called **symmetric**.

(d) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the **determinant** of $A$, denoted $\det(A)$ or $|A|$, is $ad - bc$.

(e) Assume $A$ is a square, $n \times n$, matrix.

• If there exists an $n \times n$ matrix $B$ such that $AB = BA = I_n$

  then $B$ is called the **inverse** of $A$, denoted by $A^{-1}$, and $A$ is called **invertible**

• The **$ij$th minor** of $A$ is the $(n-1) \times (n-1)$ matrix $M_{ij}$ resulting from $A$ when the $i$th row and $j$th column are removed.

• The **$ij$th cofactor** of $A$ is denoted $A_{ij}$ and defined by $A_{ij} = (-1)^{i+j} |M_{ij}|$

• The **determinant** of $A$, denoted $\det(A)$ or $|A|$, is given by $\det(A) = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$.

  • $A$ is **upper triangular** if $a_{ij} = 0$ for all $i > j$.
  
  • $A$ is **lower triangular** if $a_{ij} = 0$ for all $i < j$.
  
  • $A$ is **diagonal** if $a_{ij} = 0$ for all $i \neq j$.

2. Inverse Theorems. Assume $A$ and $B$ are both invertible matrices.

(a) $(AB)^{-1} = A^{-1}B^{-1}$

(b) $(A^T)^{-1} = (A^{-1})^T$

3. Determinant Theorems.

(a) If $A$ is a triangular matrix (upper or lower) then $\det(A) = a_{11}a_{22} \ldots a_{nn}$

(b) $\det(A^T) = \det(A)$

(c) $\det(AB) = \det(A) \det(B)$

(d) If $A$ is invertible then $\det(A^{-1}) = \frac{1}{\det(A)}$.

4. Let $A$ be an $n \times n$ matrix.

The following are equivalent (TFAE)

(a) $A$ is invertible.

(b) $\det(A) \neq 0$.

(c) $Ax = b$ has a unique solution for any $b$.

(d) $A$ is row equivalent to the identity matrix.

(e) The only solution to $Ax = 0$ is the zero vector.

(f) The reduced row echelon form of $A$ has $n$ pivots.