1. Partition the interval \([0, 2]\) into \(n\) subintervals. Let \(\Delta x_k\) be the width of the \(k\)th subinterval and let \(x_k^*\) be some point in that subinterval, then \(\sum_{k=1}^{n} \left( \sqrt{4 - (x_k^*)^2} \cdot \Delta x_k \right)\) is a Riemann sum that approximates \(\int_0^2 \sqrt{4 - x^2} \, dx\); furthermore the value of that definite integral is the limit of such Riemann sums. The region above the \(x\)-axis and below the graph of \(y = \sqrt{4 - x^2}\) over the interval \([0, 2]\) is one-quarter of a circle with radius 2, hence its area and the value of the definite integral equals \((1/4) \cdot \pi \cdot 2^2 = \pi\). (The left-hand plot below shows this region, \(n = 5\) equal-width subintervals, and rectangles with heights computed at midpoints of subintervals.)

2. Partition the interval \([5, 12]\) into \(n\) equal-width subintervals, then \(\Delta x = (12 - 5)/n = 7/n\). The first (left-hand) subinterval is \([5, 5 + \Delta x]\), the second subinterval is \([5 + \Delta x, 5 + 2 \cdot \Delta x]\); in general, the right endpoint of the \(k\)th subinterval is \(x_k = 5 + k \cdot \Delta x\). The sum contains the expression \(7 + (5 + k \cdot \Delta x)^3\), that is just \(7 + (x_k)^3\). The sum \(\sum_{k=1}^{n} (7 + (5 + k \cdot \Delta x)^3) \cdot \Delta x = \sum_{k=1}^{n} (7 + x_k^3) \cdot \Delta x\) is a Riemann sum that approximates \(\int_5^{12} (7 + x^3) \, dx\).

The right-hand plot above shows the region between the \(x\)-axis and the graph of \(y = 7 + x^3\) over the interval \([5, 12]\) together with some rectangles whose heights are computed at the right endpoint of their base subinterval.

Note: Theorem 7.4.2 (page 400) implies \(\sum_{k=1}^{n} \left( 7 + \left( 5 + k \cdot \frac{7}{n} \right)^3 \right) \cdot \frac{7}{n} = \left( \frac{7}{4} \right) \cdot \left( \frac{2901 n^2 + 3206 n + 833}{n^2} \right)\).

3. Partition the interval \([0, 6]\) into a collection of subintervals. For each subinterval, evaluate \(f\) at some point of this subinterval, multiply that function value by the width of this subinterval; finish by adding all of those products. We have a table of values for \(f(x)\) at integer values of \(x\). It is convenient to use subintervals whose endpoints are integers; there are many ways to do that. Here are several alternative approximations of \(\int_0^6 f(x) \, dx\).

- \(\Delta x = 3\), left endpoint: \(f(0) \cdot 3 + f(3) \cdot 3 = (8.8) \cdot 3 + (28.8) \cdot 3 = 26.4 + 8.4 = 34.0\)
- haphazard: \(f(2) \cdot 3 + f(4) \cdot 2 + f(6) \cdot 1 = (2.4) \cdot 3 + (4.0) \cdot 2 + (4.0) \cdot 1 = 19.2\)
- \(\Delta x = 2\), midpoint: \(f(1) \cdot 2 + f(3) \cdot 2 + f(5) \cdot 2 = (4.0) \cdot 2 + (2.8) \cdot 2 + (4.8) \cdot 2 = 8.0 + 5.6 + 9.6 = 23.2\)
- \(\Delta x = 1\), left endpoint: \(f(0) + f(1) + f(2) + f(3) + f(4) + f(5) = 8.8 + 4.0 + 2.8 + 2.8 + 4.0 + 4.8 = 26.8\)
- \(\Delta x = 1\), right endpoint: \(f(1) + f(2) + f(3) + f(4) + f(5) + f(6) = 4.0 + 2.4 + 2.8 + 4.0 + 4.8 + 4.0 = 22.0\)

4. a) \(x \cdot (1 - x)\) equals 0 at the ends of the interval \([0, 1]\), is positive in the interior of that interval, and is negative outside. The left-hand plot below shows the graph of \(y = x \cdot (1 - x)\) over \([0, 1]\).

b) Whether we rotate the region about the \(y\)-axis or the \(x\)-axis, it is moderately simple to begin by setting up a Riemann sum that involve computing what happens to a thin vertical rectangle that goes from 0 to the upper bounding curve: the height of such a rectangle is easy to compute as a function of \(x\). On the other hand, beginning with a thin horizontal rectangle seems messier: its left end computed as one function of \(x\) and its right end as another function of \(x\).

c) Suppose \(x\) is in \([0, 1]\). A thin vertical strip at \(x\), thickness \(\Delta x\) and height \(x \cdot (1 - x)\), is distance \(x\) from the \(y\)-axis; rotating this strip around the \(y\)-axis will generate a cylindrical shell with (approximate) volume
6. a) The total volume of all shells is computed by \[ \sum_{k=1}^{n} \left( 2 \pi \cdot x_k \cdot (x_k \cdot (1 - x_k)) \cdot \Delta x_k \right), \]
a Riemann sum whose limit is the definite integral \[ \int_0^1 2 \pi \cdot x \cdot (x \cdot (1 - x)) \, dx. \]

6. b) If region rotated is about \( x \)-axis, the disk method yields the definite integral \[ \int_0^1 \pi \cdot (x \cdot (1 - x))^2 \, dx. \]

5. The right-hand plot above shows graphs of the straight line \( y_1 = x + 2 \) and the parabola \( y_2 = x^2 \). Those curves meet at the points \((-1, 1)\) and \((2, 4)\); the straight line lies above the parabola over the interval \([-1, 2]\). The area of the region between those curves is computed by the definite integral \[ \int_{-1}^2 ((x + 2) - x^2) \, dx \]

7. The substitution \( x = \tan(\theta) \) implies \( dx = \sec(\theta)^2 \, d\theta \) and
\[
\int \left[ \frac{1}{(x-1) \cdot (x+1)} \right] \, dx = \int \left[ \frac{1/2}{x - 1} - \frac{1/2}{x + 1} \right] \, dx = \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + C.
\]

Intermediate steps in the preceding equation involve the identities \( 1 + \tan(\theta)^2 = \sec(\theta)^2 \) and \( \sqrt{w^2} = |w| \); the final step is a consequence of \( \theta = \arctan(x) \) and \( \sec(\theta) > 0 \) on \((-\pi/2, \pi/2)\), the range of the arctangent function.

Now use integration by parts: \( \int A \cdot dB = A \cdot B - \int B \cdot dA \). Let \( A = \sec(\theta) \) and \( dB = \sec(\theta)^2 \, d\theta \), then \( dA = \sec(\theta) \cdot \tan(\theta) \, d\theta \), \( B = \tan(\theta) \) and
\[
\int \sec(\theta)^3 \, d\theta = \int \sec(\theta) \cdot d(\tan(\theta))
\]
\[
= \sec(\theta) \cdot \tan(\theta) - \int \tan(\theta) \cdot (\sec(\theta) \cdot \tan(\theta)) \, d\theta
\]
\[
= \sec(\theta) \tan(\theta) - \int \sec(\theta) \cdot \tan(\theta)^2 \, d\theta
\]
\[
= \sec(\theta) \tan(\theta) - \int \sec(\theta) \cdot (-1 + \sec(\theta)^2) \, d\theta
\]
\[
= \sec(\theta) \tan(\theta) + \int \sec(\theta) \, d\theta - \int \sec(\theta)^3 \, d\theta.
\]

Solve the preceding equation for \( \int \sec(\theta)^3 \, d\theta \) and evaluate the remaining integral:
\[
\int \sec(\theta)^3 \, d\theta = \frac{1}{2} \left( \sec(\theta) \tan(\theta) + \int \sec(\theta) \, d\theta \right) = \frac{1}{2} \left( \sec(\theta) \tan(\theta) + \ln |\sec(\theta) + \tan(\theta)| \right) + C.
\]

Now we are ready to calculate a numerical value. The integral with respect to \( x \) over the interval \([0, 1]\) is transformed into an integral with respect to \( \theta \) over the interval \([\arctan(0), \arctan(1)] = [0, \pi/4]\). Note that \( \sec(0) = 1 \), \( \tan(0) = 0 \), \( \sec(\pi/4) = \sqrt{2} \), \( \tan(\pi/4) = 1 \) and the value of the definite integral is
\[
\frac{1}{2} \left( \sqrt{2} \cdot 1 + \ln (\sqrt{2} + 1) \right) - \frac{1}{2} \cdot (1 \cdot 0 + \ln (1 + 0)) = \frac{1}{\sqrt{2}} + \frac{1}{2} \ln (\sqrt{2} + 1).
\]
Section 8.8 provides an alternative substitution for problem 7: \( x = \sinh(t) \) implies \( dx = \cosh(t) \, dt \), \( \sqrt{1 + x^2} = \sqrt{1 + \sinh(t)^2} = \cosh(t) \), and
\[
\int \sqrt{1 + x^2} \, dx = \int \cosh(t) \cdot \left( \cosh(t) \, dt \right) = \int \cosh(t)^2 \, dt = \frac{1}{2} \left( t + \sinh(t) \cdot \cosh(t) \right) + C.
\]

8. If \( y = (1/2) \, x^2 \), then \( y' = x \) and \( \sqrt{1 + (y')^2} = \sqrt{1 + x^2} \); the definite integral \( \int_0^2 \sqrt{1 + x^2} \, dx \) computes the arclength over the interval \([0, 2]\). Calculate the numerical value by adapting the solution to the preceding problem. The definite integral with respect to \( \theta \) is over the interval \([\arctan(0), \arctan(2)] = [0, \arctan(2)]\). The identity \( \sec(\theta)^2 = \tan(\theta)^2 + 1 \) implies \( \sec(\arctan(2)) = \sqrt{5} \). The arclength is \( \frac{1}{2} \left( \sqrt{5} \cdot 2 + 2 \ln \left( \sqrt{5} + 2 \right) \right) \).

9. a) \( \int_0^1 \ln(x) \, dx \) is an improper integral because 0 is not in the domain of the natural logarithm function and \( \lim_{x \to 0^+} \ln(x) = -\infty \).
b) Let \( A = \ln(x) \) and \( dB = dx \), then \( dA = \left(1/x\right) \, dx \), \( B = x \), and
\[
\int \ln(x) \, dx = \ln(x) \cdot x - \int x \cdot d\left( \ln(x) \right) = \ln(x) \cdot x - \int x \cdot \left( \frac{1}{x} \, dx \right) = x \cdot \ln(x) - \int dx = x \cdot \ln(x) - x + C.
\]
c) \( \int_0^1 \ln(x) \, dx = \lim_{L \to 0^+} \int_L^1 \ln(x) \, dx = \lim_{L \to 0^+} (x \cdot \ln|x| - x) \bigg|_{x=L} = -1 \lim_{L \to 0^+} L \cdot \ln(L) \)
d) \( -L \cdot \ln(L) = \frac{-\ln(L)}{1/L} \) is indeterminate of type \( \frac{\infty}{\infty} \) as \( L \to 0^+ \). The derivative of the numerator is \(-1/L\), the derivative of the denominator is \(-1/L^2\) and it is never equal to 0 in \((0, 1)\), and \( \lim_{L \to 0^+} \frac{-1/L}{-1/L^2} = \lim_{L \to 0^+} L = 0 \); hence l’Hôpital’s Rule implies \( \lim_{L \to 0^+} \left( -L \cdot \ln(L) \right) = \lim_{L \to 0^+} \frac{-\ln(L)}{1/L} \) also exists and equals 0. Hence \( \int_0^1 \ln(x) \, dx = -1 \).

10. Let \( S(t) \) be the ounces of salt in the tank at time \( t \).
   a) Salt flows in and salt flows out. The net rate of change for \( S(t) \) can be decomposed:
   \[
dS = \text{rate in} - \text{rate out} = (2 \text{ gal/min}) \cdot 4 \text{ ounces gal} - (2 \text{ gal/min}) \cdot \left( \frac{S(t)}{50 \text{ gal}} \right) = 8 - \frac{1}{25} \cdot S(t).
   \]
   Since there are 25 ounces of salt in the tank at time \( t = 0 \), we have the initial condition \( S(0) = 25 \).
b) The differential equation \( S'(t) = 8 - \left(1/25\right) S(t) \) can be rewritten as \( S'(t) + \left(1/25\right) S(t) = 8 \) for which \( \mu = e^{\int \left(1/25\right) \, dt} = e^{t/25} \) is an integrating factor. Using it, we find
\[
\frac{d}{dt} \left( e^{t/25} \cdot S(t) \right) = e^{t/25} \cdot S'(t) + \frac{1}{25} e^{t/25} S(t) = 8 \cdot e^{t/25} = \frac{d}{dt} \left( 8 \cdot 25 \cdot e^{t/25} \right)
\]
and infer \( S(t) = 8 \cdot 25 + C \cdot e^{-t/25} \) where \( C \) is an arbitrary constant.
   The initial condition implies \( S(0) = 8 \cdot 25 + C \cdot e^0 = 200 + C \) and \( C = 25 - 200 = -175 \). Hence the Initial Value Problem has solution \( S(t) = 200 - 175 \cdot e^{-t/25} \) for \( t \geq 0 \).
c) \( \lim_{t \to \infty} S(t) = 200 \) ounces of salt and the limiting concentration of salt is \( \lim_{t \to \infty} \frac{S(t)}{50} = 4 \text{ ounces-per-gallon} \).
   Note that the limit of the tank’s concentration is equal to the concentration in the inflow.

11. a) \( \frac{1}{x} \) is a positive and decreasing function on \([1, \infty)\), the improper integral \( \int_1^\infty \frac{dx}{x} = \lim_{R \to \infty} \ln(R) = \infty \) diverges, therefore so does \( \sum_{k=1}^\infty \frac{1}{k} \).
b) \( \frac{5}{e^{x/2}} = 5 \cdot e^{-x/2} \) is a positive and decreasing function on \([0, \infty)\), the improper integral \( \int_0^\infty 5 \cdot e^{-x/2} \, dx = 10 - 10 \cdot \lim_{R \to \infty} e^{-R/2} = 10 \) converges, therefore so does \( \sum_{k=1}^\infty \frac{5}{e^{k/2}} \).
   This is a convergent geometric series:
\[
\sum_{k=1}^\infty \frac{5}{e^{k/2}} = 5 \cdot \frac{1}{e} \sum_{k=0}^\infty \left( \frac{1}{\sqrt{e}} \right)^k = \frac{5}{\sqrt{e}} \cdot \frac{1}{1 - 1/\sqrt{e}} = \frac{5}{\sqrt{e} - 1}.
\]
12. Begin by citing our main result about a geometric series:

\[ \sum_{n=0}^{\infty} r^n \text{ converges if and only if } |r| < 1; \quad \text{if this series converges, then its limit is } \frac{1}{1-r}. \]

Analyze \( \sum_{n=0}^{\infty} \left(x - \frac{3}{2}\right)^n \) by letting \( r = x - \frac{3}{2} \) and adapting the above statements to get ones involving \( x \).

a) The series converges at \( x = 1 \) because \(|1 - 3/2| = 1/2 < 1\); the series diverges at \( x = 3 \) because \(|3 - 3/2| = 3/2 > 1\).

b) The series converges if and only if \(|x - 3/2| < 1\); that inequality is equivalent to \(3/2 - 1 < x < 3/2 + 1\). The interval of convergence for this series is \((1/2, 5/2)\).

c) If \( w \) is in the interval of convergence, then

\[ \sum_{n=0}^{\infty} \left(w - \frac{3}{2}\right)^n = \frac{1}{1 - (w - 3/2)} = \frac{2}{5 - 2w}. \]

13. Alternating Series Test: If \( \{A_k\}_{k=1}^{\infty} \) is a sequence of positive terms which (a) decrease and (b) tend to 0 as \( k \to \infty \), then \( \sum_{k=1}^{\infty} (-1)^{k-1} A_k \) converges.

Suppose \( A_k = \frac{1}{k} \). If \( k \) is positive, then \( k + 1 > k \) implies \( A_k = \frac{1}{k} > \frac{1}{k+1} = A_{k+1} \). Furthermore, \( \lim_{k \to \infty} A_k = \lim_{k \to \infty} \frac{1}{k} = 0 \). Therefore \( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \) converges.

14. a) \( \frac{3^k}{k!} \) is positive for each positive integer \( k \) and \( \lim_{k \to \infty} \frac{3^{k+1}/(k+1)!}{3^k/k!} = \lim_{k \to \infty} \frac{3^{k+1}/3^k}{(k+1)!/(k!)} = \lim_{k \to \infty} \frac{3}{k+1} = 0; \)

since that limit is less than 1, the Limit Ratio Test implies \( \sum_{k=1}^{\infty} \frac{3^k}{k!} \) converges.

b) \( \lim_{k \to \infty} \frac{1}{5 + 4^{-k}} = \frac{1}{5} + 0 = \frac{1}{5} \) is not equal to zero, therefore \( \sum_{k=1}^{\infty} \frac{1}{5 + 4^{-k}} \) does not converge.

15. A function \( f \) and its \( n^{\text{th}} \)-order Taylor polynomial \( P_n \) centered at \( a \) have the same function and derivative values at \( a \), up to order \( n \): i.e., \( f^{(k)}(a) = P_n^{(k)}(a) \) for all integers \( k \) between 0 and \( n \). If \( q(x) = 1 - x + 2x^2 \) is the second-order Maclaurin polynomial for \( f \), then \( f(0) = q(0) = 1 \) so we can exclude the left-most plot which shows a curve through the origin. \( f'(0) = q'(0) = 1 \) so we can exclude the second plot from the left which shows a curve with positive slope as it crosses the \( y\)-axis. \( f''(0) = q''(0) = 4 \) is positive suggesting the graph of \( f \) is concave up in some neighborhood of 0, that would lead to rejection of the third plot which is concave down there and choosing the right-most plot which is concave up around \( x = 0 \).

16. a) \( \sin' = \cos, \sin'' = -\sin, \sin''' = -\cos, \sin^{(4)} = \sin, \text{ and } \sin^{(k)} = \sin^{(k-4)} \) for all integers \( k \geq 5 \). Since \( \cos(\pi/2) = 0 \), we can infer \( \sin^{(odd)}(\pi/2) = 0 \); for the even-order derivatives, we have \( \sin^{(2k)}(\pi/2) = (-1)^k \sin(\pi/2) = (-1)^k \).

The fourth-order Taylor polynomial for \( \sin \) centered at \( \pi/2 \) is \( P_4(x) = 1 - \frac{1}{2} \cdot \left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24} \cdot \left(x - \frac{\pi}{2}\right)^4 \).

b) The answer to part (a) is a fourth-order Taylor polynomial. The fifth derivative of \( \sin \) is \( \cos \), the maximum value of \( |\cos| \) on the interval \( [0, \pi] \) is 1. If \( x \) is in \( [0, \pi] \), then \( |x - \pi/2| \leq \pi/2 \). Hence the Remainder Bound Theorem (11.9.3 on page 677) implies that if \( x \) is in \( [0, \pi] \), then \( |\sin(x) - P_4(x)| \leq \frac{1}{5!} \cdot \left(\frac{\pi}{2}\right)^5 \leq \frac{\pi^5}{3840} \approx 0.0797. \)

We can get a smaller bound. Since odd-order derivatives of sine have value 0 at \( \pi/2 \), we can also regard the answer to part (a) as the fifth-order Taylor polynomial and compute the bound \( \frac{1}{6!} \cdot \left(\frac{\pi}{2}\right)^6 = \frac{\pi^6}{46080} \approx 0.0209. \)

c) \( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k} \) is the Taylor series for \( \sin(x) \) centered at \( \pi/2 \).

d-e) If \( x = \pi/2 \), then the series in part (c) is \( 1 + 0 + 0 + \cdots + 0 + \cdots \), clearly convergent. If \( x \neq \pi/2 \), then none of the terms in that series is zero and their absolute values are positive — therefore the Limit Ratio Test is
18. Use some facts about geometric series. The Limit Ratio Test implies the above series is absolutely convergent for all \( x \) because
\[
\lim_{k \to \infty} \frac{|x - \pi/2|^{2(k+1)} / (2(k+1))!}{|x - \pi/2|^{2k} / (2k)!} = \lim_{k \to \infty} \frac{|x - \pi/2|^{2}}{(2k+1) \cdot (2k+2)} = 0.
\]
Series converges implies individual terms have limit zero, i.e., we now know \( \lim_{k \to \infty} \frac{|x - \pi/2|^{2k}}{(2k)!} = 0 \) for all \( x \), hence the Remainder Bound has limit 0 for every \( x \) and this series must converge to \( \sin(x) \).

19. a) Even-order derivatives of \( \sin(x) \) are \( \pm \sin(x) \) whose value at \( x = 0 \) is 0; \( \sin^{2k+1}(x) = (-1)^k \cdot \cos(x) \) whose values at \( x = 0 \) are \( (-1)^k \). Hence \( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1} \) is the Maclaurin series for \( \sin(x) \). The ideas used above in solution for 16 (d-e) are easily adapted to show this series converges to \( \sin(x) \) for all \( x \).

b) Odd-order derivatives of \( \cos(x) \) are \( \pm \sin(x) \) whose value at \( x = 0 \) is 0; \( \cos(2k)(x) = (-1)^k \cdot \cos(x) \) whose values at \( x = 0 \) are \( (-1)^k \). Hence \( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \cdot x^{2k} \) is the Maclaurin series for \( \cos(x) \). The ideas used above in solution for 16 (d-e) are easily adapted to show this series converges to \( \cos(x) \) for all \( x \).

c) The answers to parts (a) and (b) together with Theorem 11.10.2 (page 687) imply
\[
\frac{d}{dx} \sin(x) = \frac{d}{dx} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1} \right) = \sum_{k=0}^{\infty} \frac{d}{dx} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot (2k+1) \cdot x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \cdot \frac{2k+1}{2k+1} \cdot x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \cdot x^{2k} = \cos(x).
\]
d) The answers to parts (a) and (b) together with Theorem 11.10.4 (page 689) imply
\[
\int \sin(x) \, dx = \int \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1} \right) \, dx = \sum_{k=0}^{\infty} \int \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1} \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot \frac{x^{2k+2}}{2k+2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)(k+1)!} \cdot x^{2k+1} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j)!} \cdot x^{2j} = -1 \cdot \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \cdot x^{2j} = - \cos(x) + C.
\]

18. Use some facts about geometric series. \( t \) in \( (-1, 1) \) implies \( \frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = \sum_{k=0}^{\infty} (-t^2)^k = \sum_{k=0}^{\infty} (-1)^k \cdot t^{2k} \).

If \( x \) is in \( (-1, 1) \), then Theorem 11.10.4 (page 689) implies
\[
\arctan(x) = \int_{0}^{x} \frac{1}{1+t^2} \, dt = \int_{0}^{x} \left( \sum_{k=0}^{\infty} (-1)^k \cdot t^{2k} \right) \, dt = \sum_{k=0}^{\infty} (-1)^k \left( \int_{0}^{x} t^{2k} \, dt \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cdot x^{2k+1}
\]
The fifth-degree term of that Maclaurin series is \( (1/5) \cdot x^5 \), its fifth-derivative is 24 and that equals \( \arctan'(0) \).

19. a) \( (x-2)^2 + y^2 = 2^2; \) center at \((2, 0)_R\) with radius 2.  Plot of circle is shown below in answer to part (c).

b) If \( r = 4 \cos(\theta) \), then \( r'(\theta) = \frac{d}{d\theta} \left( 4 \cos(\theta) \right) = -4 \sin(\theta) \). Then \( x = r(\theta) \cdot \cos(\theta) \) and \( y = r(\theta) \cdot \sin(\theta) \) imply
\[
\frac{dx}{d\theta} = -4 \sin(\theta) \cdot \cos(\theta) + 4 \cos(\theta) \cdot (-\sin(\theta)) = -8 \sin(\theta) \cdot \cos(\theta) = -4 \sin(2\theta)
\]
\[
\frac{dy}{d\theta} = -4 \sin(\theta) \cdot \sin(\theta) + 4 \cos(\theta) \cdot \cos(\theta) = 4 \cdot \left( \cos(\theta)^2 - \sin(\theta)^2 \right) = 4 \cos(2\theta).
\]
If \( \frac{dx}{d\theta} \neq 0 \), then the point on the curve has a non-vertical tangent whose slope is

\[
\frac{dy}{dx} = \frac{4 \cdot (\cos(\theta)^2 - \sin(\theta)^2)}{-8 \sin(\theta) \cdot \cos(\theta)} = \frac{1}{2} \cdot \frac{\sin(\theta)^2 - \cos(\theta)^2}{\sin(\theta) \cdot \cos(\theta)} = \frac{1}{2} \cdot \left( \tan(\theta) - \cot(\theta) \right) \]

The alternative expressions for \( x'(\theta) \) and \( y'(\theta) \) yield a corresponding one for slope:

\[
\frac{dy}{dx} = 4 \cos(2\theta) - 4 \sin(2\theta) = -\cot(2\theta).
\]

c) \( \sin(\pi/4) = 1/\sqrt{2} = \cos(\pi/4) \) and the slope of the tangent at the corresponding point is 0. \( \sin(\pi/6) = 1/2, \cos(\pi/6) = \sqrt{3}/2 \) and the slope of the tangent at the corresponding point is \(-1/\sqrt{3}\).

The following plot shows the circle, lines through the origin with \( \theta = \pi/6 \) and \( \theta = \pi/4 \), together with the lines tangent to the circle at the corresponding points on the circle.

d) The circle is traversed as \( \theta \) varies in any interval of length \( \pi \).

(It is acceptable to answer by just mentioning one such interval, e.g., \([0, \pi]\).)

e) Formula 12.2.2 (page 717) implies the length of this closed curve is

\[
\int_0^\pi \sqrt{\left(4 \cos(\theta)\right)^2 + \left(-4 \sin(\theta)\right)^2} \, d\theta = \int_0^\pi \sqrt{16 \cos(\theta)^2 + 16 \sin(\theta)^2} \, d\theta
\]

\[
= \int_0^\pi 4 \cdot \sqrt{\cos(\theta)^2 + \sin(\theta)^2} \, d\theta = \int_0^\pi 4 \cdot \sqrt{1} \, d\theta = 4 \theta \bigg|_0^\pi = 4 \pi.
\]

f) Formula 12.3.2 (page 721) implies the area of the region enclosed by this curve is

\[
\int_0^\pi \frac{1}{2} \left( 4 \cos(\theta) \right)^2 \, d\theta = 4 \cdot \int_0^\pi 2 \cos(\theta)^2 \, d\theta = 4 \cdot \left( \theta + \sin(\theta) \cdot \cos(\theta) \right) \bigg|_0^\pi = 4 \pi.
\]