Vector Spaces: 3.1

- A set is a collection of objects. Usually the objects in a set share some common features. For example the set of all $m$ by $n$ matrices can be given a name such as $M_{mn}$ and described by

$$M_{mn} = \{ A : A \text{ is an } m \times n \text{ matrix} \}$$

This is stated: "The set $M_{\text{sub} m n}$ is the set of all $A$ such that $A$ is an $m$ by $n$ matrix'.

- The symbol $\in$ means "is a member of". For example $A \in M_{22}$ is stated "$A$ is a member of $M_{22}$".

- The symbol $\subseteq$ means "is a subset of". If $A$ and $B$ are sets and every member of $A$ is also a member of $B$ then we say $A \subseteq B$.

- A real vector space $V$ is a set of objects, called vectors together with two operations called addition and scalar multiplication that satisfy the following axioms (Numbering Scheme based on Olsavsky).

1. Addition is a binary operation on $V$ which is both commutative and associative.
   - (a) If $x \in V$ and $y \in V$ then $x + y \in V$. closure under addition
   - (b) For all $x$, $y$, and $z$ in $V$, $(x + y) + z = x + (y + z)$. associative property of addition
   - (c) If $x$ and $y$ are in $V$, then $x + y = y + x$. commutative property of addition
2. There is a vector $0 \in V$ such that $x + 0 = x$ for all $x \in V$. additive identity
3. If $x \in V$ then there is a vector $-x$ such that $x + (-x) = 0$. additive inverse
4. If $x \in V$ and $\alpha$ is a scalar (real number), then $\alpha x \in V$. closure under scalar multiplication
5. Scalar Multiplication Properties.
   - (a) If $x$ and $y$ are in $V$ and $\alpha$ is a scalar, then $\alpha(x + y) = \alpha x + \alpha y$. distributive property 1
   - (b) If $x \in V$ and $\alpha$ and $\beta$ are scalars, then $(\alpha + \beta)x = \alpha x + \beta x$. distributive property 2
   - (c) If $x \in V$ and $\alpha$ and $\beta$ are scalars, then $\alpha(\beta x) = (\alpha \beta)x$. associative law of scalar mult.
   - (d) For every vector $x \in V$, $1 x = x$. scalar multiplicative identity

- Determine whether the following sets are vector spaces.
  1. $V$ is the set of all 2 by 2 upper triangular matrices.
  2. $V$ the set of all functions which are continuous on $[0, 2]$.
  3. $V$ is the set of all polynomials of the form $p(x) = 3 + \alpha x$, where $\alpha \in \mathbb{R}$.
  4. $V$ is the set of all polynomials of degree 2 or less.

- Classic examples of vector spaces.
  1. $\mathbb{R}^n = \text{the set of all } n\text{-vectors.}$
  2. $M_{mn} = \text{the set of all } m \times n \text{ matrices.}$
  3. $C[a, b] = \text{the set of all continuous functions on } [a, b].$
  4. $C(-\infty, \infty) = \text{the set of all continuous functions on } \mathbb{R}.$
  5. $P_n = \text{the set of all polynomials of degree } \leq n.$

- Important Theorem for Later. If $V$ is a vector space then
  1. $0x = 0$ Proof: $0x = (0 + 0)x \Rightarrow 0 = 0x$
  2. $(-1)x = -x$ Proof: $(-1)x = (0 - 1)x \Rightarrow (-1)x = 0 - x \Rightarrow x + (-1)x = 0$