System Analysis - Rank and Nullity

- Recall Example One from Geometry of Solutions

$$\begin{array}{ccccccccccc} x_1 & - & x_2 & + & x_3 & + & 2x_4 & - & x_5 & = & 1 \\ & & x_2 & + & 6x_3 & + & 2x_4 & + & x_5 & = & 0 \\ & & x_1 & + & 7x_3 & + & 5x_4 & + & 2x_5 & = & 3 \end{array}$$

  Augmented matrix form:

$$\left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 1 \\ 0 & 1 & 6 & 2 & 1 & 0 \\ 1 & 0 & 7 & 5 & 2 & 3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 1 \\ 0 & 1 & 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{array}\right)$$

  and our infinite set of solutions were

$$X = \{(7t - 7s - 7,\ 3t - 6s - 4,\ s,\ -2t + 2,\ t) : \quad s, t \in \mathcal{R}\}$$

  or

$$X = (7t - 7s,\ 3t - 6s,\ s,\ -2t,\ t) + (-7, -4, 0, 2, 0) = X_h + X_p$$

- Definitions: We say that for the above example there are two **free variables**, s and t, and three **dependent variables**, x_1, x_2, and x_3.

- Observations regarding the coefficient matrix:

$$\begin{pmatrix} 1 & -1 & 1 & 2 & -1 \\ 0 & 1 & 6 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

  - There are three columns with leading ones = number of dependent variables.
  - There are two columns without leading ones = number of free variables.

- Definitions: Given an m by n matrix (m-rows, n-columns) and let A be the row echelon form of this matrix.

  - **rank** (A) = number of leading ones in A = number of dependent variables in the solution of the system.
  - **nullity** $(A) = n$ - rank(A) = number of free variables in the solution set of the system.

- Observation: An m by n consistent system of equations will have a unique solution if and only if the nullity of the coefficient matrix is zero.

- Definitions

  - A system with fewer equations than unknowns (m¡n) is called **underdetermined**.
    An underdetermined system is usually consistent and dependent (infinitely many solutions).
  - A system with more equations than unknowns (m¿ n) is called **overdetermined**.
    An overdetermined system is usually inconsistent (no solutions).

- Example 1: Consider the system below. Is it underdetermined or overdetermined? What is the rank and nullity of the coefficient matrix? Describe the geometry of the solution. (point, line, plane, or hyperplane)

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & = & 1 \\ & & x_2 & + & 2x_3 & + & 4x_4 & = & 5 \end{array}$$

- Example 2: Consider the system below. Is it underdetermined, overdetermined, or neither? What is the rank and nullity of the coefficient matrix? Determine the conditions on $B = (a, b, c)$ so that the system has a solution. Will the solution be unique under these conditions.

$$\begin{array}{rcrcrcl} & & x_1 & + & x_3 & = & a \\ 2x_1 & + & x_2 & + & 4x_3 & = & b \\ 3x_1 & + & x_2 & + & 5x_3 & = & c \end{array}$$