Inverses

1. **Definitions:** Assume $A$ is a square, $n \times n$, matrix.
   If there exists an $n \times n$ matrix $B$ such that
   \[ AB = BA = I_n \]
   then $B$ is called the **inverse** of $A$, denoted by $A^{-1}$, and $A$ is called **invertible**. In this case $A$ is also called **nonsingular**. If $A$ does not have an inverse it is called singular.

2. **Inverse of a 2 by 2 matrix**

   If \[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \] then \[ A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \]

   Notice: The number $ad - bc$ determines whether $A$ has an inverse or not. This number is called the **determinant**.

3. **Observation.** If an $n$ by $n$ system of equations is expressed as
   \[ Ax = b \]
   where $A$ is an invertible matrix, then the solution is
   \[ x = A^{-1}b \]

4. Solve the system of equations using the inverse of the matrix.
   \[
   \begin{align*}
   x_1 + 2x_2 &= 5 \\
   3x_1 + 4x_2 &= 6
   \end{align*}
   \]

5. **Inverse Theorems.** Assume $A$ and $B$ are both invertible matrices.

   (a) \( (AB)^{-1} = B^{-1}A^{-1} \)
   (b) \( (A^{-1})^{-1} = A \)
   (c) \( (A^T)^{-1} = (A^{-1})^T \)
   (d) \( (kA)^{-1} = \frac{1}{k}A^{-1} \)

6. **Getting the inverse of larger matrices.**

   You want to find $X$ such that
   \[ AX = I_n \]

   Set up the augmented matrix $[A|I_n]$ and perform row operations until $A$ is equivalent to $I_n$ then the right half is the inverse of $A$. In other words convert $[A|I_n]$ to $[I_n|A^{-1}]$ using row operations (if you can).
7. Example: Find the inverse of \[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 12 \\
2 & 7 & 5
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 2 & 3 & | & 1 & 0 & 0 \\
4 & 9 & 12 & | & 0 & 1 & 0 \\
2 & 7 & 5 & | & 0 & 0 & 1
\end{pmatrix}
\]

8. **Know This**

Let \(A\) be an \(n \times n\) matrix.

The following are equivalent (TFAE)

(a) \(A\) is invertible (nonsingular or has an inverse).
(b) \(A \sim I_n\).
(c) The reduced row echelon form of \(A\) has \(n\) pivots.
(d) \(\text{rank}(A) = n\).
(e) \(\text{nullity}(A) = 0\).
(f) \(Ax = b\) has a unique solution for any \(b\).
(g) The only solution to \(Ax = 0\) is \(x = 0\).