4.5 UNDETERMINED COEFFICIENTS—ANNIHILATOR APPROACH

**REVIEW MATERIAL**
- Review Theorems 4.1.6 and 4.1.7 (Section 4.1)

**INTRODUCTION** We saw in Section 4.1 that an nth-order differential equation can be written

\[ a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 D y + a_0 y = g(x), \]  

where \( D^k y = \frac{d^k y}{dx^k}, \) \( k = 0, 1, \ldots, n. \) When it suits our purpose, (1) is also written as \( L(y) = g(x), \) where \( L \) denotes the linear nth-order differential, or polynomial, operator

\[ a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0. \]  

Not only is the operator notation a helpful shorthand, but also on a very practical level the application of differential operators enables us to justify the somewhat mind-numbing rules for determining the form of particular solution \( y_p \) that were presented in the preceding section. In this section there are no special rules; the form of \( y_p \) follows almost automatically once we have found an appropriate linear differential operator that annihilates \( g(x) \) in (1). Before investigating how this is done, we need to examine two concepts.

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**Factoring Operators** When the coefficients \( a_i, i = 0, 1, \ldots, n \) are real constants, a linear differential operator (1) can be factored whenever the characteristic polynomial \( a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 \) factors. In other words, if \( r_1 \) is a root of the auxiliary equation

\[ a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0, \]

then \( L = (D - r_1) P(D), \) where the polynomial expression \( P(D) \) is a linear differential operator of order \( n - 1. \) For example, if we treat \( D \) as an algebraic quantity, then the operator \( D^2 + 5D + 6 \) can be factored as \( (D + 2)(D + 3) \) or as \( (D + 3)(D + 2). \) Thus if a function \( y = f(x) \) possesses a second derivative, then

\[ (D^2 + 5D + 6)y = (D + 2)(D + 3)y = (D + 3)(D + 2)y. \]

This illustrates a general property:

**Factors of a linear differential operator with constant coefficients commute**

A differential equation such as \( y'' + 4y' + 4y = 0 \) can be written as

\[ (D^2 + 4D + 4)y = 0 \quad \text{or} \quad (D + 2)(D + 2)y = 0 \quad \text{or} \quad (D + 2)^2 y = 0. \]

**Annihilator Operator** If \( L \) is a linear differential operator with constant coefficients and \( f \) is a sufficiently differentiable function such that

\[ L(f(x)) = 0, \]

then \( L \) is said to be an annihilator of the function. For example, a constant function \( y = k \) is annihilated by \( D, \) since \( Dk = 0. \) The function \( y = x \) is annihilated by the differential operator \( D^2 \) since the first and second derivatives of \( x \) are 1 and 0, respectively. Similarly, \( D^3x^2 = 0, \) and so on.

The differential operator \( D^6 \) annihilates each of the functions

\[ 1, \ x, \ x^2, \ \ldots, \ x^{n-1}. \]
As an immediate consequence of (3) and the fact that differentiation can be done term by term, a polynomial
\[ c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} \]  
(4)
can be annihilated by finding an operator that annihilates the highest power of \( x \).

The functions that are annihilated by a linear \( n \)-th order differential operator \( L \) are simply those functions that can be obtained from the general solution of the homogeneous differential equation \( L(y) = 0 \).

The differential operator \((D - \alpha)^n\) annihilates each of the functions
\[ e^{\alpha x}, \quad xe^{\alpha x}, \quad x^2e^{\alpha x}, \quad \ldots, \quad x^{n-1}e^{\alpha x}. \]  
(5)

To see this, note that the auxiliary equation of the homogeneous equation \((D - \alpha)^n y = 0\) is \((m - \alpha)^n = 0\). Since \( \alpha \) is a root of multiplicity \( n \), the general solution is
\[ y = c_1e^{\alpha x} + c_2xe^{\alpha x} + \cdots + c_ne^{\alpha x}. \]  
(6)

**EXAMPLE 1** Annihilator Operators

Find a differential operator that annihilates the given function.
(a) \( 1 - 5x^2 + 8x^3 \)  
(b) \( e^{-3x} \)  
(c) \( 4e^{2x} - 10xe^{2x} \)

**SOLUTION** (a) From (3) we know that \( D^2x^3 = 0 \), so it follows from (4) that
\[ D^2(1 - 5x^2 + 8x^3) = 0. \]

(b) From (5), with \( \alpha = -3 \) and \( n = 1 \), we see that
\[ (D + 3)e^{-3x} = 0. \]

(c) From (5) and (6), with \( \alpha = 2 \) and \( n = 2 \), we have
\[ (D - 2)^2(4e^{2x} - 10xe^{2x}) = 0. \]

When \( \alpha \) and \( \beta \) \( \beta \) \( \beta \) \( \beta \) are real numbers, the quadratic formula reveals that \([m^2 - 2\alpha m + (\alpha^2 + \beta^2)]^n = 0\) has complex roots \( \alpha + i\beta \), \( \alpha - i\beta \), both of multiplicity \( n \). From the discussion at the end of Section 4.3 we have the next result.

The differential operator \([D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n\) annihilates each of the functions
\[ e^{\alpha x} \cos \beta x, \quad xe^{\alpha x} \cos \beta x, \quad x^2e^{\alpha x} \cos \beta x, \quad \ldots, \quad x^{n-1}e^{\alpha x} \cos \beta x, \]
\[ e^{\alpha x} \sin \beta x, \quad xe^{\alpha x} \sin \beta x, \quad x^2e^{\alpha x} \sin \beta x, \quad \ldots, \quad x^{n-1}e^{\alpha x} \sin \beta x. \]  
(7)

**EXAMPLE 2** Annihilator Operator

Find a differential operator that annihilates \( 5e^{-x} \cos 2x - 9e^{-x} \sin 2x \).

**SOLUTION** Inspection of the functions \( e^{-x} \cos 2x \) and \( e^{-x} \sin 2x \) shows that \( \alpha = -1 \) and \( \beta = 2 \). Hence from (7) we conclude that \( D^2 + 2D + 5 \) will annihilate each function. Since \( D^2 + 2D + 5 \) is a linear operator, it will annihilate any linear combination of these functions such as \( 5e^{-x} \cos 2x - 9e^{-x} \sin 2x \).
When $\alpha = 0$ and $n = 1$, a special case of (7) is

$$
(D^2 + \beta^2) \left[ \cos \beta x \sin \beta x \right] = 0.
$$

(8)

For example, $D^2 + 16$ will annihilate any linear combination of $\sin 4x$ and $\cos 4x$.

We are often interested in annihilating the sum of two or more functions. As we have just seen in Examples 1 and 2, if $L$ is a linear differential operator such that $L(y_1) = 0$ and $L(y_2) = 0$, then $L$ will annihilate the linear combination $c_1y_1(x) + c_2y_2(x)$. This is a direct consequence of Theorem 4.1.2. Let us now suppose that $L_1$ and $L_2$ are linear differential operators with constant coefficients such that $L_1$ annihilates $y_1(x)$ and $L_2$ annihilates $y_2(x)$, but $L_1(y_2) \neq 0$ and $L_2(y_1) \neq 0$. Then the product of differential operators $L_1L_2$ annihilates the sum $c_1y_1(x) + c_2y_2(x)$. We can easily demonstrate this, using linearity and the fact that $L_1L_2 = L_2L_1$:

$$
L_1L_2(y_1 + y_2) = L_1L_2(y_1) + L_1L_2(y_2)
= L_2L_1(y_1) + L_1L_2(y_2)
= L_2[L_1(y_1)] + L_1[L_2(y_2)] = 0.
$$

For example, we know from (3) that $D^2$ annihilates $7 - x$ and from (8) that $D^2 + 16$ annihilates $\sin 4x$. Therefore the product of operators $D^2(D^2 + 16)$ will annihilate the linear combination $7 - x + 6 \sin 4x$.

**Note** The differential operator that annihilates a function is not unique. We saw in part (b) of Example 1 that $D + 3$ will annihilate $e^{-3x}$, but so will differential operators of higher order as long as $D + 3$ is one of the factors of the operator. For example, $(D + 3)(D + 1)$, $(D + 3)^2$, and $D^3(D + 3)$ all annihilate $e^{-3x}$. (Verify this.) As a matter of course, when we seek a differential annihilator for a function $y = f(x)$, we want the operator of lowest possible order that does the job.

**Undetermined Coefficient** This brings us to the point of the preceding discussion. Suppose that $L(y) = g(x)$ is a linear differential equation with constant coefficients and that the input $g(x)$ consists of finite sums and products of the functions listed in (3), (5), and (7)—that is, $g(x)$ is a linear combination of functions of the form

$$
 k \text{ (constant)}, \quad x^m, \quad x^m e^{\alpha x}, \quad x^m e^{\alpha x} \cos \beta x, \quad \text{and} \quad x^m e^{\alpha x} \sin \beta x,
$$

where $m$ is a nonnegative integer and $\alpha$ and $\beta$ are real numbers. We now know that such a function $g(x)$ can be annihilated by a differential operator $L_1$ of lowest order, consisting of a product of the operators $D^n$, $(D - \alpha)^n$, and $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^n$. Applying $L_1$ to both sides of the equation $L(y) = g(x)$ yields $L_1L(y) = L_1(g(x)) = 0$. By solving the homogenous higher-order equation $L_1L(y) = 0$, we can discover the form of a particular solution $y_p$ for the original nonhomogeneous equation $L(y) = g(x)$. We then substitute this assumed form into $L(y) = g(x)$ to find an explicit particular solution. This procedure for determining $y_p$, called the method of undetermined coefficients is illustrated in the next several examples.

Before proceeding, recall that the general solution of a nonhomogeneous linear differential equation $L(y) = g(x)$ is $y = y_c + y_p$, where $y_c$ is the complementary function—that is, the general solution of the associated homogeneous equation $L(y) = 0$. The general solution of each equation $L(y) = g(x)$ is defined on the interval $(-\infty, \infty)$. 

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EXAMPLE 3  General Solution Using Undetermined Coefficient

Solve \( y'' + 3y' + 2y = 4x^2 \). \( (9) \)

**SOLUTION**  Step 1. First, we solve the homogeneous equation \( y'' + 3y' + 2y = 0 \). Then, from the auxiliary equation \( m^2 + 3m + 2 = (m + 1)(m + 2) = 0 \) we find \( m_1 = -1 \) and \( m_2 = -2 \), and so the complementary function is

\[ y_c = c_1 e^{-x} + c_2 e^{-2x}. \]

Step 2. Now, since \( 4x^2 \) is annihilated by the differential operator \( D^2 \), we see that \( D^2(D^2 + 3D + 2)y = 4D^3x^2 \) is the same as

\[ D^3(D^2 + 3D + 2)y = 0. \]  \( (10) \)

The auxiliary equation of the fifth-order equation in (10)

\[ m^5(m^2 + 3m + 2) = 0 \quad \text{or} \quad m^5(m + 1)(m + 2) = 0, \]

has roots \( m_1 = m_2 = m_3 = 0 \), \( m_4 = -1 \), and \( m_5 = -2 \). Thus its general solution must be

\[ y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + c_5 e^{-2x}. \]  \( (11) \)

The terms in the shaded box in (11) constitute the complementary function of the original equation (9). We can then argue that a particular solution \( y_p \) of (9) should also satisfy equation (10). This means that the terms remaining in (11) must be the basic form of \( y_p \):

\[ y_p = A + Bx + Cx^2, \]  \( (12) \)

where, for convenience, we have replaced \( c_1, c_2, \) and \( c_3 \) by \( A, B, \) and \( C \), respectively. For (12) to be a particular solution of (9), it is necessary to find specific coefficient \( A, B, \) and \( C \). Differentiating (12), we have

\[ y_p' = B + 2Cx, \quad y_p'' = 2C, \]

and substitution into (9) then gives

\[ y_p'' + 3y_p' + 2y_p = 2C + 3B + 6Cx + 2A + 2Bx + 2Cx^2 = 4x^2. \]

Because the last equation is supposed to be an identity, the coefficients of like powers of \( x \) must be equal:

\[
\begin{align*}
2C & \quad x^2 + 2B + 6C & \quad x + 2A + 3B + 2C & \quad 4x^2 + 0x + 0.
\end{align*}
\]

That is \( 2C = 4, \quad 2B + 6C = 0, \quad 2A + 3B + 2C = 0 \). \( (13) \)

Solving the equations in (13) gives \( A = 7, \quad B = -6, \) and \( C = 2 \). Thus \( y_p = 7 - 6x + 2x^2 \).

Step 3. The general solution of the equation in (9) is \( y = y_c + y_p \) or

\[ y = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2. \]
EXAMPLE 4  General Solution Using Undetermined Coefficient

Solve \( y'' - 3y' = 8e^{3x} + 4 \sin x \).  \hfill (14)

**SOLUTION**  

**Step 1.** The auxiliary equation for the associated homogeneous equation \( y'' - 3y' = 0 \) is \( m^2 - 3m = m(m - 3) = 0 \), so \( y_c = c_1 + c_2 e^{3x} \).

**Step 2.** Now, since \( (D - 3)e^{3x} = 0 \) and \( (D^2 + 1) \sin x = 0 \), we apply the differential operator \( (D - 3)(D^2 + 1) \) to both sides of (14):

\[
(D - 3)(D^2 + 1)(D^2 - 3D)y = 0.
\]  \hfill (15)

The auxiliary equation of (15) is

\[
(m - 3)(m + 1)(m - 3) = 0 \quad \text{or} \quad m(m - 3)^2 + 1 = 0.
\]

Thus

\[
y = c_1 + c_2 e^{3x} + c_3 x e^{3x} + c_4 \cos x + c_5 \sin x.
\]

After excluding the linear combination of terms in the box that corresponds to \( y_c \), we arrive at the form of \( y_p \):

\[
y_p = A x e^{3x} + B \cos x + C \sin x.
\]

Substituting \( y_p \) in (14) and simplifying yield

\[
y''_p - 3y'_p = 3A e^{3x} + (-B - 3C) \cos x + (3B - C) \sin x = 8e^{3x} + 4 \sin x.
\]

Equating coefficients gives \( 3A = 8 \), \(-B - 3C = 0 \), and \(3B - C = 4 \). We find \( A = \frac{8}{3}, B = \frac{2}{5}, \) and \( C = -\frac{2}{5} \), and consequently,

\[
y_p = \frac{8}{3} x e^{3x} + \frac{6}{5} \cos x - \frac{2}{5} \sin x.
\]

**Step 3.** The general solution of (14) is then

\[
y = c_1 + c_2 e^{3x} + \frac{8}{3} x e^{3x} + \frac{6}{5} \cos x - \frac{2}{5} \sin x.
\]

EXAMPLE 5  General Solution Using Undetermined Coefficient

Solve \( y'' + y = x \cos x - \cos x \).  \hfill (16)

**SOLUTION**  

The complementary function is \( y_c = c_1 \cos x + c_2 \sin x \). Now by comparing \( \cos x \) and \( \cos x \) with the functions in the first row of (7), we see that \( \alpha = -1 \) and \( n = 1 \), and so \( (D^2 + 1)^2 \) is an annihilator for the right-hand member of the equation in (16). Applying this operator to the differential equation gives

\[
(D^2 + 1)^2 (D^2 + 1)y = 0 \quad \text{or} \quad (D^2 + 1)^3 y = 0.
\]

Since \( i \) and \(-i \) are both complex roots of multiplicity 3 of the auxiliary equation of the last differential equation, we conclude that

\[
y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x + c_5 x^2 \cos x + c_6 x^2 \sin x.
\]

We substitute

\[
y_p = A x \cos x + B x \sin x + C x^2 \cos x + E x^2 \sin x
\]

into (16) and simplify:

\[
y''_p + y_p = 4Ex \cos x - 4Cx \sin x + (2B + 2C) \cos x + (-2A + 2E) \sin x = x \cos x - \cos x.
\]
Equating coefficients gives the equations $4E = 1$, $-4C = 0$, $2B + 2C = -1$, and $-2A + 2E = 0$, from which we find $A = \frac{1}{4}$, $B = -\frac{1}{2}$, $C = 0$, and $E = \frac{1}{2}$. Hence the general solution of (16) is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{4} x \cos x - \frac{1}{2} x \sin x + \frac{1}{4} x^2 \sin x.$$  

\[ \square \]

**EXAMPLE 6**  Form of a Particular Solution

Determine the form of a particular solution for

$$y'' - 2y' + y = 10e^{-2x} \cos x.$$  

**SOLUTION**  The complementary function for the given equation is $y_c = c_1 e^\alpha + c_2 xe^\alpha$.

Now from (7), with $\alpha = -2$, $\beta = 1$, and $n = 1$, we know that

$$(D^2 + 4D + 5)e^{-2x} \cos x = 0.$$  

Applying the operator $D^2 + 4D + 5$ to (17) gives

$$(D^2 + 4D + 5)(D^2 - 2D + 1)y = 0.$$  

Since the roots of the auxiliary equation of (18) are $-2 - i$, $-2 + i$, 1, and 1, we see from

$$y = c_1 e^{\alpha x} + c_2 xe^{\alpha x} + c_3 e^{-2x} \cos x + c_4 e^{-2x} \sin x$$

that a particular solution of (17) can be found with the form

$$y_p = Ae^{-2x} \cos x + Be^{-2x} \sin x.$$  

\[ \square \]

**EXAMPLE 7**  Form of a Particular Solution

Determine the form of a particular solution for

$$y''' - 4y'' + 4y' = 5x^2 - 6x + 4x^2 e^{2x} + 3e^{3x}.$$  

**SOLUTION**  Observe that

$$D^3(5x^2 - 6x) = 0, \quad (D - 2)^3x^2 e^{2x} = 0,$$

and

$$(D - 5)e^{3x} = 0.$$  

Therefore $D^3(D - 2)^3(D - 5)$ applied to (19) gives

$$D^3(D - 2)^3(D - 5)(D^2 - 4D^2 + 4D)y = 0$$

or

$$D^3(D - 2)^3(D - 5)y = 0.$$  

The roots of the auxiliary equation for the last differential equation are easily seen to be 0, 0, 0, 2, 2, 2, 2, 2, and 5. Hence

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{2x} + c_6 xe^{2x} + c_7 x^2 e^{2x} + c_8 x^3 e^{2x} + c_9 x^4 e^{2x} + c_{10} x^5 e^{2x}.$$  

(20)

Because the linear combination $c_1 + c_5 e^{2x} + c_6 xe^{2x}$ corresponds to the complementary function of (19), the remaining terms in (20) give the form of a particular solution of the differential equation:

$$y_p = Ax + Bx^2 + Cx^3 + Ex^2 e^{2x} + Fx^3 e^{2x} + Gx^4 e^{2x} + H x^5 e^{2x}.$$  

\[ \square \]

**Summary of the Method**  For your convenience the method of undetermined coefficients is summarized as follows
UNDETERMINED COEFFICIENTS—ANNIHILATOR APPROACH

The differential equation \( L(y) = g(x) \) has constant coefficients, and the function \( g(x) \) consists of finite sums and products of constants, polynomials, exponential functions \( e^{ax} \), sines, and cosines.

(i) Find the complementary solution \( y_c \) for the homogeneous equation \( L(y) = 0 \).
(ii) Operate on both sides of the nonhomogeneous equation \( L(y) = g(x) \) with a differential operator \( L \) that annihilates the function \( g(x) \).
(iii) Find the general solution of the higher-order homogeneous differential equation \( L(L(y)) = 0 \).
(iv) Delete from the solution in step (iii) all those terms that are duplicated in the complementary solution \( y_c \) found in step (i). Form a linear combination \( y_p \) of the terms that remain. This is the form of a particular solution of \( L(y) = g(x) \).
(v) Substitute \( y_p \) found in step (iv) into \( L(y) = g(x) \). Match coefficient of the various functions on each side of the equality, and solve the resulting system of equations for the unknown coefficients in \( y_p \).
(vi) With the particular solution found in step (v), form the general solution \( y = y_c + y_p \) of the given differential equation.

REMARKS

The method of undetermined coefficients is not applicable to linear differential equations with variable coefficients nor is it applicable to linear equations with constant coefficients when \( g(x) \) is a function such as

\[
\begin{align*}
g(x) &= \ln x, \\
g(x) &= \frac{1}{x}, \\
g(x) &= \tan x, \\
g(x) &= \sin^{-1} x,
\end{align*}
\]

and so on. Differential equations in which the input \( g(x) \) is a function of this last kind will be considered in the next section.

EXERCISES 4.5

In Problems 1 – 10 write the given differential equation in the form \( L(y) = g(x) \), where \( L \) is a linear differential operator with constant coefficients. If possible, factor \( L \).

1. \( 9y'' - 4y = \sin x \)
2. \( y'' - 5y = x^2 - 2x \)
3. \( y'' - 4y' - 12y = x - 6 \)
4. \( 2y'' - 3y' - 2y = 1 \)
5. \( y''' + 10y'' + 25y' = e^x \)
6. \( y''' + 4y' = e^x \cos 2x \)
7. \( y'' + 2y' - 13y' + 10y = xe^{-x} \)
8. \( y'' + 4y' + 3y' = x^2 \cos x - 3x \)
9. \( y^{(4)} + 8y' = 4 \)
10. \( y^{(4)} - 8y'' + 16y = (x^3 - 2x)e^{4x} \)

In Problems 11 – 14 verify that the given differential operator annihilates the indicated functions.

11. \( D^4; \ y = 10x^3 - 2x \)
12. \( 2D - 1; \ y = 4e^{x/2} \)
13. \( (D^2 - 2)(D + 5); \ y = e^{2x} + 3e^{-5x} \)
14. \( D^4 + 64; \ y = 2 \cos 8x - 5 \sin 8x \)

In Problems 15 – 26 find a linear differential operator that annihilates the given function.

15. \( 1 + 6x - 2x^3 \)
16. \( x^3(1 - 5x) \)
17. \( 1 + 7e^{2x} \)
18. \( x + 3xe^{6x} \)
19. \( \cos 2x \)
20. \( 1 + \sin x \)
21. \( 13x + 9x^2 - \sin 4x \)
22. \( 8x - \sin x + 10 \cos 5x \)
23. \( e^{-x} + 2xe^{x} - x^2e^{x} \)
24. \( (2 - e^t)^2 \)
25. \( 3 + e^x \cos 2x \)
26. \( e^{-x} \sin x - e^{2x} \cos x \)

Answers to selected odd-numbered problems begin on page ANS-5.
In Problems 27–34 find linearly independent functions that are annihilated by the given differential operator.

27. \( D^4 \)
28. \( D^2 + 4D \)
29. \((D - 6)(2D + 3)\)
30. \(D^2 - 9D - 36 \)
31. \(D^2 + 5 \)
32. \(D^2 - 6D + 10 \)
33. \(D^3 - 10D^2 + 25D \)
34. \(D^2(D - 5)(D - 7) \)

In Problems 35–64 solve the given differential equation by undetermined coefficients

35. \(y'' - 9y = 54 \)
36. \(2y'' - 7y' + 5y = -29 \)
37. \(y'' + y' = 3 \)
38. \(y''' + 2y'' + y' = 10 \)
39. \(y'' + 4y' + 4y = 2x + 6 \)
40. \(y'' + 3y' = 4x - 5 \)
41. \(y'' + y'' = 8x^2 \)
42. \(y'' - 2y' + y = x^3 + 4x \)
43. \(y'' - y' - 12y = e^{4x} \)
44. \(y'' + 2y' + 2y = 5e^{6x} \)
45. \(y'' - 2y' - 3y = 4e^{4x} - 9 \)
46. \(y'' + 6y' + 8y = 3e^{-2x} + 2x \)
47. \(y'' + 25y = 6 \sin x \)
48. \(y'' + 4y = 4 \cos x + 3 \sin x - 8 \)
49. \(y'' + 6y' + 9y = -xe^{4x} \)
50. \(y'' + 3y' - 10y = x(e^{x} + 1) \)
51. \(y'' - y = x^2e^{x} + 5 \)
52. \(y'' + 2y' + y = x^2e^{-x} \)
53. \(y'' - 2y' + 5y = e^{x} \sin x \)
54. \(y'' + y' + \frac{1}{4}y = e^{x} \sin 3x - \cos 3x \)

55. \(y'' + 25y = 20 \sin 5x \)
56. \(y'' + y = 4 \cos x - \sin x \)
57. \(y'' + y' + y = x \sin x \)
58. \(y'' + 4y = \cos^2 x \)
59. \(y'' + 8y'' - y = 6x^2 + 9x + 2 \)
60. \(y'' - y' + y = xe^x - e^{-x} + 7 \)
61. \(y'' - 3y' + 3y = e^x - x + 16 \)
62. \(2y'' - 3y' - 3y + 2y = (e^x + e^{-x})^2 \)
63. \(y^{(4)} - 2y'' + y = e^x + 1 \)
64. \(y^{(4)} - 4y'' = 5x^2 - e^{2x} \)

In Problems 65–72 solve the given initial-value problem.

65. \(y'' - 64y = 16, \quad y(0) = 1, \quad y'(0) = 0 \)
66. \(y'' + y' = x, \quad y(0) = 1, \quad y'(0) = 0 \)
67. \(y'' - 5y' = x - 2, \quad y(0) = 0, \quad y'(0) = 2 \)
68. \(y'' + 5y' - 6y = 10e^{2x}, \quad y(0) = 1, \quad y'(0) = 1 \)
69. \(y'' + y = 8 \cos 2x - 4 \sin x, \quad y(\pi/2) = -1, \quad y'(\pi/2) = 0 \)
70. \(y'' - 2y'' + y' = xe^x + 5, \quad y(0) = 2, \quad y'(0) = 2, \quad y''(0) = -1 \)
71. \(y'' - 4y' + 8y = x^3, \quad y(0) = 2, \quad y'(0) = 4 \)
72. \(y^{(4)} - y'' = x + e^x, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y''''(0) = 0 \)

**Discussion Problems**

73. Suppose \(L\) is a linear differential operator that factors but has variable coefficients. Do the factors of \(L\) commute? Defend your answer.

### 4.6 VARIATION OF PARAMETERS

**REVIEW MATERIAL**
- Basic integration formulas and techniques from calculus
- Review Section 2.3

**INTRODUCTION** We pointed out in the discussions in Sections 4.4 and 4.5 that the method of undetermined coefficients has two inherent weaknesses that limit its wider application to linear equations: the DE must have constant coefficients and the input function \(g(x)\) must be of the type listed in Table 4.4.1. In this section we examine a method for determining a particular solution \(y_p\) of a nonhomogeneous linear DE that has, in theory, no such restrictions on it. This method, due to the eminent astronomer and mathematician Joseph Louis Lagrange (1736–1813), is known as variation of parameters.

Before examining this powerful method for higher-order equations we revisit the solution of linear first-order differential equations that have been put into standard form. The discussion under the first heading in this section is optional and is intended to motivate the main discussion of this section that starts under the second heading. If pressed for time this motivational material could be assigned for reading.