The history of mathematics is rife with stories of people who devoted much of their lives to solving equations—algebraic equations at first and then eventually differential equations. In Sections 2.2–2.5 we will study some of the more important analytical methods for solving first-order DEs. However, before we start solving anything, you should be aware of two facts: It is possible for a differential equation to have no solutions, and a differential equation can possess solutions, yet there might not exist any analytical method for solving it. In Sections 2.1 and 2.6 we do not solve any DEs but show how to glean information about solutions directly from the equation itself. In Section 2.1 we see how the DE yields qualitative information about graphs that enables us to sketch renditions of solution curves. In Section 2.6 we use the differential equation to construct a procedure, called a numerical method, for approximating solutions.
2.1 SOLUTION CURVES WITHOUT A SOLUTION

REVIEW MATERIAL
- The first derivative as slope of a tangent line
- The algebraic sign of the first derivative indicates increasing or decreasing

INTRODUCTION Let us imagine for the moment that we have in front of us a first-order differential equation \( \frac{dy}{dx} = f(x, y) \), and let us further imagine that we can neither find nor invent a method for solving it analytically. This is not as bad a predicament as one might think, since the differential equation itself can sometimes “tell” us specifics about how its solutions “behave.”

We begin our study of first-order differential equations with two ways of analyzing a DE qualitatively. Both these ways enable us to determine, in an approximate sense, what a solution curve must look like without actually solving the equation.

2.1.1 DIRECTION FIELDS

Some Fundamental Questions We saw in Section 1.2 that whenever \( f(x, y) \) and \( \frac{df}{dy} \) satisfy certain continuity conditions, qualitative questions about existence and uniqueness of solutions can be answered. In this section we shall see that other qualitative questions about properties of solutions—How does a solution behave near a certain point? How does a solution behave as \( x \to \infty? \)—can often be answered when the function \( f \) depends solely on the variable \( y \). We begin, however, with a simple concept from calculus:

A derivative \( \frac{dy}{dx} \) of a differentiable function \( y = y(x) \) gives slopes of tangent lines at points on its graph.

Slope Because a solution \( y = y(x) \) of a first-order differential equation

\[
\frac{dy}{dx} = f(x, y)
\]

is necessarily a differentiable function on its interval \( I \) of definition, it must also be continuous on \( I \). Thus the corresponding solution curve on \( I \) must have no breaks and must possess a tangent line at each point \((x, y(x))\). The function \( f \) in the normal form (1) is called the slope function or rate function. The slope of the tangent line at \((x, y(x))\) on a solution curve is the value of the first derivative \( dy/dx \) at this point, and we know from (1) that this is the value of the slope function \( f(x, y(x)) \). Now suppose that \((x, y)\) represents any point in a region of the \( xy \)-plane over which the function \( f \) is defined. The value \( f(x, y) \) of the function \( f \) that the function \( f \) assigns to the point represents the slope of a line or, as we shall envision it, a line segment called a lineal element. For example, consider the equation \( dy/dx = 0.2xy \), where \( f(x, y) = 0.2xy \). At, say, the point \((2, 3)\) the slope of a lineal element is \( f(2, 3) = 0.2(2)(3) = 1.2 \). Figure 2.1.1(a) shows a line segment with slope 1.2 passing through \((2, 3)\). As shown in Figure 2.1.1(b), if a solution curve also passes through the point \((2, 3)\), it does so tangent to this line segment; in other words, the lineal element is a miniature tangent line at that point.

Direction Field If we systematically evaluate \( f \) over a rectangular grid of points in the \( xy \)-plane and draw a line element at each point \((x, y)\) of the grid with slope \( f(x, y) \), then the collection of all these line elements is called a direction field or a slope field of the differential equation \( dy/dx = f(x, y) \). Visually, the direction field suggests the appearance or shape of a family of solution curves of the differential equation, and consequently, it may be possible to see at a glance certain qualitative aspects of the solutions—regions in the plane, for example, in which a
solution exhibits an unusual behavior. A single solution curve that passes through a
direction field must follow the flow pattern of the field; it is tangent to a lineal element
when it intersects a point in the grid. Figure 2.1.2 shows a computer-generated direction field of the differential equation \( dy/dx = \sin(x + y) \) over a region of the xy-plane. Note how the three solution curves shown in color follow the flow of the field.

**EXAMPLE 1**

**Direction Field**

The direction field for the differential equation \( dy/dx = 0.2xy \) shown in Figure 2.1.3(a) was obtained by using computer software in which a \( 5 \times 5 \) grid of points \((mh, nh)\), \(m\) and \(n\) integers, was defined by letting \(-5 \leq m \leq 5, -5 \leq n \leq 5\), and \(h = 1\). Notice in Figure 2.1.3(a) that at any point along the \(x\)-axis \((y = 0)\) and the \(y\)-axis \((x = 0)\), the slopes are \(f(x, 0) = 0\) and \(f(0, y) = 0\), respectively, so the lineal elements are horizontal. Moreover, observe in the first quadrant that for a fixed value of \(x\) the values of \(f(x, y) = 0.2xy\) increase as \(y\) increases; similarly, for a fixed \(y\) the values of \(f(x, y) = 0.2xy\) increase as \(x\) increases. This means that as both \(x\) and \(y\) increase, the lineal elements almost become vertical and have positive slope \(f(x, y) = 0.2xy > 0\) for \(x > 0, y > 0\). In the second quadrant, \(f(x, y)\) increases as \(|x|\) and \(y\) increase, so the lineal elements again become almost vertical but this time have negative slope \(f(x, y) = 0.2xy < 0\) for \(x > 0, y > 0\). Reading from left to right, imagine a solution curve that starts at a point in the second quadrant, moves steeply downward, becomes flat as it passes through the \(y\)-axis, and then, as it enters the first quadrant, moves steeply upward—in other words, its shape would be concave upward and similar to a horseshoe. From this it could be surmised that \(y \to \infty\) as \(x \to \pm \infty\). Now in the third and fourth quadrants, since \(f(x, y) = 0.2xy > 0\) and \(f(x, y) = 0.2xy < 0\), respectively, the situation is reversed: A solution curve increases and then decreases as we move from left to right. We saw in (1) of Section 1.1 that \(y = e^{0.1x^2}\) is an explicit solution of the differential equation \(dy/dx = 0.2xy\); you should verify that a one-parameter family of solutions of the same equation is given by \(y = ce^{0.1x^2}\). For purposes of comparison with Figure 2.1.3(a) some representative graphs of members of this family are shown in Figure 2.1.3(b).

**EXAMPLE 2**

**Direction Field**

Use a direction field to sketch an approximate solution curve for the initial-value problem \(dy/dx = \sin y, y(0) = -\frac{\pi}{2}\).

**SOLUTION** Before proceeding, recall that from the continuity of \(f(x, y) = \sin y\) and \(df/dy = \cos y\), Theorem 1.2.1 guarantees the existence of a unique solution curve passing through any specific point \((x_0, y_0)\) in the plane. Now we set our computer software again for a \(5 \times 5\) rectangular region and specify (because of the initial condition) points in that region with vertical and horizontal separation of \(\frac{1}{2}\) unit—that is, at points \((mh, nh)\), \(h = 1, m\) and \(n\) integers such that \(-10 \leq m \leq 10, -10 \leq n \leq 10\). The result is shown in Figure 2.1.4. Because the right-hand side of \(dy/dx = \sin y\) is 0 at \(y = 0\), and at \(y = -\pi\), the lineal elements are horizontal at all points whose second coordinates are \(y = 0\) or \(y = -\pi\). It makes sense then that a solution curve passing through the initial point \((0, -\frac{\pi}{2})\) has the shape shown in the figure.

**Increasing/Decreasing** Interpretation of the derivative \(dy/dx\) as a function that gives slope plays the key role in the construction of a direction field. Another telling property of the first derivative will be used next, namely, if \(dy/dx > 0\) (or \(dy/dx < 0\)) for all \(x\) in an interval \(I\), then a differentiable function \(y = y(x)\) is increasing (or decreasing) on \(I\).
### Autonomous First-Order DEs

In Section 1.1 we divided the class of ordinary differential equations into two types: linear and nonlinear. We now consider briefly another kind of classification of ordinary differential equations, a classification that is of particular importance in the qualitative investigation of differential equations. An ordinary differential equation in which the independent variable does not appear explicitly is said to be autonomous. If the symbol $x$ denotes the independent variable, then an autonomous first-order differential equation can be written as

$$\frac{dy}{dx} = f(y).$$

We shall assume throughout that the function $f$ in (2) and its derivative $f'$ are continuous functions of $y$ on some interval $I$. The first-order equation

$$\frac{dy}{dx} = 1 + y^2 \quad \text{and} \quad \frac{dy}{dx} = 0.2xy$$

are autonomous and nonautonomous, respectively.

Many differential equations encountered in applications or equations that are models of physical laws that do not change over time are autonomous. As we have already seen in Section 1.3, in an applied context, symbols other than $y$ and $x$ are routinely used to represent the dependent and independent variables. For example, if $t$ represents time then inspection of

$$\frac{dA}{dt} = kA, \quad \frac{dx}{dt} = kx(n + 1 - x), \quad \frac{dT}{dt} = k(T - T_m), \quad \frac{dA}{dt} = 6 - \frac{1}{100}A,$$

where $k$, $n$, and $T_m$ are constants, shows that each equation is time independent. Indeed, all of the first-order differential equations introduced in Section 1.3 are time independent and so are autonomous.

#### Critical Points

The zeros of the function $f$ in (2) are of special importance. We say that a real number $c$ is a critical point of the autonomous differential equation (2) if it is a zero of $f$—that is, $f(c) = 0$. A critical point is also called an equilibrium point or stationary point. Now observe that if we substitute the constant function $y(x) = c$ into (2), then both sides of the equation are zero. This means:

**If $c$ is a critical point of (2), then $y(x) = c$ is a constant solution of the autonomous differential equation.**

A constant solution $y(x) = c$ of (2) is called an equilibrium solution; equilibria are the only constant solutions of (2).
As was already mentioned, we can tell when a nonconstant solution \( y = y(x) \) of (2) is increasing or decreasing by determining the algebraic sign of the derivative \( dy/dx \); in the case of (2) we do this by identifying intervals on the \( y \)-axis over which the function \( f(y) \) is positive or negative.

### Example 3 An Autonomous DE

The differential equation

\[
\frac{dP}{dt} = P(a - bP),
\]

where \( a \) and \( b \) are positive constants, has the normal form \( dP/dt = f(P) \), which is (2) with \( t \) and \( P \) playing the parts of \( x \) and \( y \), respectively, and hence is autonomous. From \( f(P) = P(a - bP) = 0 \) we see that 0 and \( a/b \) are critical points of the equation, so the equilibrium solutions are \( P(t) = 0 \) and \( P(t) = a/b \). By putting the critical points on a vertical line, we divide the line into three intervals defined by \(-\infty < P < 0, 0 < P < a/b, a/b < P < \infty\). The arrows on the line shown in Figure 2.1.5 indicate the algebraic sign of \( f(P) = P(a - bP) \) on these intervals and whether a nonconstant solution \( P(t) \) is increasing or decreasing on an interval. The following table explains the figure.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Sign of ( f(P) )</th>
<th>( P(t) )</th>
<th>Arrow</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 0))</td>
<td>minus</td>
<td>decreasing</td>
<td>points down</td>
</tr>
<tr>
<td>((0, a/b))</td>
<td>plus</td>
<td>increasing</td>
<td>points up</td>
</tr>
<tr>
<td>((a/b, \infty))</td>
<td>minus</td>
<td>decreasing</td>
<td>points down</td>
</tr>
</tbody>
</table>

Figure 2.1.5 is called a **one-dimensional phase portrait**, or simply **phase portrait**, of the differential equation \( dP/dt = P(a - bP) \). The vertical line is called a **phase line**.

**Solution Curves** Without solving an autonomous differential equation, we can usually say a great deal about its solution curves. Since the function \( f \) in (2) is independent of the variable \( x \), we may consider \( f \) defined for \(-\infty < x < \infty\) or for \( 0 \leq x < \infty\). Also, since \( f \) and its derivative \( f' \) are continuous functions of \( y \) on some interval \( I \) of the \( y \)-axis, the fundamental results of Theorem 1.2.1 hold in some horizontal strip or region \( R \) in the \( xy \)-plane corresponding to \( I \), and so through any point \((x_0, y_0)\) in \( R \) there passes only one solution curve of (2). See Figure 2.1.6(a). For the sake of discussion, let us suppose that (2) possesses exactly two critical points \( c_1 \) and \( c_2 \) and that \( c_1 < c_2 \). The graphs of the equilibrium solutions \( y(x) = c_1 \) and \( y(x) = c_2 \) are horizontal lines, and these lines partition the region \( R \) into three subregions \( R_1, R_2, \) and \( R_3 \), as illustrated in Figure 2.1.6(b). Without proof here are some conclusions that we can draw about a nonconstant solution \( y(x) \) of (2):

- If \((x_0, y_0)\) is in a subregion \( R_i, i = 1, 2, 3, \) and \( y(x) \) is a solution whose graph passes through this point, then \( y(x) \) remains in the subregion \( R_i \) for all \( x \). As illustrated in Figure 2.1.6(b), the solution \( y(x) \) in \( R_2 \) is bounded below by \( c_1 \) and above by \( c_2 \), that is, \( c_1 < y(x) < c_2 \) for all \( x \). The solution curve stays within \( R_2 \) for all \( x \) because the graph of a nonconstant solution of (2) cannot cross the graph of either equilibrium solution \( y(x) = c_1 \) or \( y(x) = c_2 \). See Problem 33 in Exercises 2.1.
- By continuity of \( f \) we must then have either \( f(y) > 0 \) or \( f(y) < 0 \) for all \( x \) in a subregion \( R_i, i = 1, 2, 3 \). In other words, \( f(y) \) cannot change signs in a subregion. See Problem 33 in Exercises 2.1.
Since \( dy/dx = f(y(x)) \) is either positive or negative in a subregion \( R_i \), \( i = 1, 2, 3 \), a solution \( y(x) \) is strictly monotonic—that is, \( y(x) \) is either increasing or decreasing in the subregion \( R_i \). Therefore \( y(x) \) cannot be oscillatory, nor can it have a relative extremum (maximum or minimum). See Problem 33 in Exercises 2.1.

If \( y(x) \) is bounded above by a critical point \( c_1 \) (as in subregion \( R_1 \) where \( y(x) - c_1 \) for all \( x \)), then the graph of \( y(x) \) must approach the graph of the equilibrium solution \( y(x) = c_1 \) either as \( x \to \infty \) or as \( x \to -\infty \). If \( y(x) \) is bounded— that is, bounded above and below by two consecutive critical points (as in subregion \( R_2 \) where \( c_1 < y(x) < c_2 \) for all \( x \))—then the graph of \( y(x) \) must approach the graphs of the equilibrium solutions \( y(x) = c_1 \) and \( y(x) = c_2 \), one as \( x \to \infty \) and the other as \( x \to -\infty \). If \( y(x) \) is bounded below by a critical point (as in subregion \( R_3 \) where \( c_2 < y(x) < c_3 \) for all \( x \)), then the graph of \( y(x) \) must approach the graph of the equilibrium solution \( y(x) = c_2 \) either as \( x \to \infty \) or as \( x \to -\infty \). See Problem 34 in Exercises 2.1.

With the foregoing facts in mind, let us reexamine the differential equation in Example 3.

### Example 4 Example 3 Revisited

The three intervals determined on the \( P \)-axis or phase line by the critical points 0 and \( a/b \) now correspond in the \( tP \)-plane to three subregions defined by:

- \( R_1: -\infty < P < 0 \),  \( R_2: 0 < P < a/b \), and \( R_3: a/b < P < \infty \),

where \(-\infty < t < \infty \). The phase portrait in Figure 2.1.7 tells us that \( P(t) \) is decreasing in \( R_1 \), increasing in \( R_2 \), and decreasing in \( R_3 \). If \( P(0) = P_0 \) is an initial value, then in \( R_1, R_2, \) and \( R_3 \) we have, respectively, the following:

(i) \( P_0 < 0 \), \( P(t) \) is bounded above. Since \( P(t) \) is decreasing, \( P(t) \) decreases without bound for increasing \( t \), and so \( P(t) \to 0 \) as \( t \to -\infty \).

This means that the negative \( t \)-axis, the graph of the equilibrium solution \( P(t) = 0 \), is a horizontal asymptote for a solution curve.

(ii) For \( 0 < P_0 < a/b \), \( P(t) \) is bounded. Since \( P(t) \) is increasing, \( P(t) \to a/b \) as \( t \to \infty \) and \( P(t) \to 0 \) as \( t \to -\infty \). The graphs of the two equilibrium solutions, \( P(t) = 0 \) and \( P(t) = a/b \), are horizontal lines that are horizontal asymptotes for any solution curve starting in this subregion.

(iii) For \( P_0 > a/b \), \( P(t) \) is bounded below. Since \( P(t) \) is decreasing, \( P(t) \to a/b \) as \( t \to \infty \). The graph of the equilibrium solution \( P(t) = a/b \) is a horizontal asymptote for a solution curve.

In Figure 2.1.7 the phase line is the \( P \)-axis in the \( tP \)-plane. For clarity the original phase line from Figure 2.1.5 is reproduced to the left of the plane in which the subregions \( R_1, R_2, \) and \( R_3 \) are shaded. The graphs of the equilibrium solutions \( P(t) = a/b \) and \( P(t) = 0 \) (the \( t \)-axis) are shown in the figure as blue dashed lines; the solid graphs represent typical graphs of \( P(t) \) illustrating the three cases just discussed.

In a subregion such as \( R_2 \) in Example 4, where \( P(t) \) is decreasing and unbounded below, we must necessarily have \( P(t) \to -\infty \). Do not interpret this last statement to mean \( P(t) \to -\infty \) as \( t \to \infty \); we could have \( P(t) \to -\infty \) as \( t \to T \), where \( T > 0 \) is a finite number that depends on the initial condition \( P(t_0) = P_0 \). Thinking in dynamic terms, \( P(t) \) could “blow up” in finite time; thinking graphically, \( P(t) \) could have a vertical asymptote at \( t = T > 0 \). A similar remark holds for the subregion \( R_3 \).

The differential equation \( dy/dx = \sin y \) in Example 2 is autonomous and has an infinite number of critical points, since \( \sin y = 0 \) at \( y = n\pi, n \) an integer. Moreover, we now know that because the solution \( y(x) \) that passes through \((0, -\frac{\pi}{2})\) is bounded
above and below by two consecutive critical points \((-\pi, y(x) = 0)\) and is decreasing \((\sin y = 0 \text{ for } -\pi < y < 0)\), the graph of \(y(x)\) must approach the graphs of the equilibrium solutions as horizontal asymptotes: \(y(x) \to -\pi \text{ as } x \to \infty\) and \(y(x) \to 0 \text{ as } x \to -\infty\).

### Example 5  
**Solution Curves of an Autonomous DE**

The autonomous equation \(dy/dx = (y - 1)^2\) possesses the single critical point 1. From the phase portrait in Figure 2.1.8(a) we conclude that a solution \(y(x)\) is an increasing function in the subregions defined by \(-\infty < y < 1\) and \(1 < y < \infty\), where \(-\infty < x < \infty\). For an initial condition \(y(0) = y_0 < 1\), a solution \(y(x)\) is increasing and bounded above by 1, and so \(y(x) \to 1\) as \(x \to \infty\); for \(y(0) = y_0 > 1\) a solution \(y(x)\) is increasing and unbounded.

Now \(y(x) = 1 - 1/(x + c)\) is a one-parameter family of solutions of the differential equation. (See Problem 4 in Exercises 2.2.) A given initial condition determines a value for \(c\). For the initial conditions, say, \(y(0) = -1 < 1\) and \(y(0) = 2 > 1\), we find, in turn, that \(y(x) = 1 - 1/(x + 1/2)\), and \(y(x) = 1 - 1/(x - 1)\). As shown in Figures 2.1.8(b) and 2.1.8(c), the graph of each of these rational functions possesses a vertical asymptote. But bear in mind that the solutions of the IVPs

\[
\frac{dy}{dx} = (y - 1)^2, \quad y(0) = -1 \quad \text{and} \quad \frac{dy}{dx} = (y - 1)^2, \quad y(0) = 2
\]

are defined on special intervals. They are, respectively,

\[
y(x) = 1 - \frac{1}{x + \frac{1}{2}}, \quad -\frac{1}{2} < x < \infty \quad \text{and} \quad y(x) = 1 - \frac{1}{x - 1}, \quad -\infty < x < 1.
\]

The solution curves are the portions of the graphs in Figures 2.1.8(b) and 2.1.8(c) shown in blue. As predicted by the phase portrait, for the solution curve in Figure 2.1.8(b), \(y(x) \to 1\) as \(x \to \infty\); for the solution curve in Figure 2.1.8(c), \(y(x) \to \infty\) as \(x \to 1\) from the left.

#### Attractors and Repellers  
Suppose that \(y(x)\) is a nonconstant solution of the autonomous differential equation given in (1) and that \(c\) is a critical point of the DE. There are basically three types of behavior that \(y(x)\) can exhibit near \(c\). In Figure 2.1.9 we have placed \(c\) on four vertical phase lines. When both arrowheads on either side of the dot labeled \(c\) point toward \(c\), as in Figure 2.1.9(a), all solutions \(y(x)\) of (1) that start from an initial point \((x_0, y_0)\) sufficiently near \(c\) exhibit the asymptotic behavior \(\lim_{x \to \infty} y(x) = c\). For this reason the critical point \(c\) is said to be
asymptotically stable. Using a physical analogy, a solution that starts near \( c \) is like a charged particle that, over time, is drawn to a particle of opposite charge, and so \( c \) is also referred to as an attractor. When both arrowheads on either side of the dot labeled \( c \) point away from \( c \), as in Figure 2.1.9(b), all solutions \( y(x) \) of (1) that start from an initial point \((x_0, y_0)\) move away from \( c \) as \( x \) increases. In this case the critical point \( c \) is said to be unstable. An unstable critical point is also called a repeller, for obvious reasons. The critical point \( c \) illustrated in Figures 2.1.9(c) and 2.1.9(d) is neither an attractor nor a repeller. But since \( c \) exhibits characteristics of both an attractor and a repeller—that is, a solution starting from an initial point \((x_0, y_0)\) sufficiently near \( c \) is attracted to \( c \) from one side and repelled from the other side—we say that the critical point \( c \) is semi-stable. In Example 3 the critical point \( a/b \) is asymptotically stable (an attractor) and the critical point 0 is unstable (a repeller). The critical point 1 in Example 5 is semi-stable.

### Autonomous DEs and Direction Fields

If a first-order differential equation is autonomous, then we see from the right-hand side of its normal form \( dy/dx = f(y) \) that slopes of lineal elements through points in the rectangular grid used to construct a direction field for the DE depend solely on the \( y \)-coordinate of the points. Put another way, lineal elements passing through points on any horizontal line must all have the same slope and therefore are parallel; slopes of lineal elements along any vertical line will, of course, vary. These facts are apparent from inspection of the horizontal yellow strip and vertical blue strip in Figure 2.1.10. The figure exhibits a direction field for the autonomous equation \( dy/dx = 2(y - 1) \). The red lineal elements in Figure 2.1.10 have zero slope because they lie along the graph of the equilibrium solution \( y = 1 \).

#### Translation Property

You may recall from precalculus mathematics that the graph of a function \( y = f(x - k) \), where \( k \) is a constant, is the graph of \( y = f(x) \) rigidly translated or shifted horizontally along the \( x \)-axis by an amount \( |k| \); the translation is to the right if \( k > 0 \) and to the left if \( k < 0 \). It turns out that under the conditions stipulated for (2), solution curves of an autonomous first-order DE are related by the concept of translation. To see this, let’s consider the differential equation \( dy/dx = y(3 - y) \), which is a special case of the autonomous equation considered in Examples 3 and 4. Because \( y = 0 \) and \( y = 3 \) are equilibrium solutions of the DE, their graphs divide the \( xy \)-plane into three subregions \( R_1, R_2, \) and \( R_3 \):

\[
R_1: \quad -\infty \quad y \quad 0, \quad R_2: \quad 0 \quad y \quad 3, \quad \text{and} \quad R_3: \quad 3 \quad y \quad \infty.
\]

In Figure 2.1.11 we have superimposed on a direction field of the DE six solutions curves. The figure illustrates that all solution curves of the same color, that is, solution curves lying within a particular subregion \( R_k \), all look alike. This is no coincidence but is a natural consequence of the fact that lineal elements passing through points on any horizontal line are parallel. That said, the following translation property of an autonomous DE should make sense:

If \( y(x) \) is a solution of an autonomous differential equation \( dy/dx = f(y) \), then \( y(x) = y(x - k), \ k \ a \ constant, \) is also a solution.

Thus, if \( y(x) \) is a solution of the initial-value problem \( dy/dx = f(y), \ y(0) = y_0 \), then \( y_1(x) = y(x - x_0) \) is a solution of the IVP \( dy/dx = f(y), \ y(x_0) = y_0 \). For example, it is easy to verify that \( y(x) = e^x, \ x < -\infty \), is a solution of the IVP \( dy/dx = y, \ y(0) = 1 \) and so a solution \( y_1(x) \) of, say, \( dy/dx = y, \ y(5) = 1 \) is \( y(x) = e^x \) translated 5 units to the right:

\[
y_1(x) = y(x - 5) = e^{x - 5}, \quad x < -\infty.
\]
2.1 SOLUTION CURVES WITHOUT A SOLUTION

EXERCISES 2.1

2.1.1 DIRECTION FIELDS

In Problems 1–4 reproduce the given computer-generated direction field. Then sketch, by hand, an approximate solution curve that passes through each of the indicated points. Use different colored pencils for each solution curve.

1. \( \frac{dy}{dx} = x^2 - y^2 \)
   (a) \( y(-2) = 1 \)     (b) \( y(3) = 0 \)
   (c) \( y(0) = 2 \)     (d) \( y(0) = 0 \)

2. \( \frac{dy}{dx} = e^{-0.01x^2} \)
   (a) \( y(-6) = 0 \)     (b) \( y(0) = 1 \)
   (c) \( y(0) = -4 \)     (d) \( y(8) = -4 \)

3. \( \frac{dy}{dx} = 1 - xy \)
   (a) \( y(0) = 0 \)     (b) \( y(-1) = 0 \)
   (c) \( y(2) = 2 \)     (d) \( y(0) = -4 \)

4. \( \frac{dy}{dx} = (\sin x) \cos y \)
   (a) \( y(0) = 1 \)     (b) \( y(1) = 0 \)
   (c) \( y(3) = 3 \)     (d) \( y(0) = \frac{5}{2} \)

In Problems 5–12 use computer software to obtain a direction field for the given differential equation. By hand, sketch an approximate solution curve passing through each of the given points.

5. \( y' = x \)
   (a) \( y(0) = 0 \)
   (b) \( y(0) = -3 \)

6. \( y' = x + y \)
   (a) \( y(-2) = 2 \)
   (b) \( y(1) = -3 \)

7. \( \frac{dy}{dx} = -x \)
   (a) \( y(1) = 1 \)
   (b) \( y(0) = 4 \)

8. \( \frac{dy}{dx} = \frac{1}{y} \)
   (a) \( y(0) = 1 \)
   (b) \( y(2) = -1 \)

9. \( \frac{dy}{dx} = 0.2x^2 + y \)
   (a) \( y(0) = \frac{1}{2} \)
   (b) \( y(2) = -1 \)

10. \( \frac{dy}{dx} = xe^y \)
    (a) \( y(0) = -2 \)
    (b) \( y(1) = 2.5 \)
11. \( y' = y - \cos \frac{x}{2} \)

12. \( \frac{dy}{dx} = 1 - \frac{y}{x} \)

(a) \( y(2) = 2 \)

(b) \( y(-1) = 0 \)

In Problems 13 and 14 the given figure represents the graph of \( f(y) \) and \( f(x) \), respectively. By hand, sketch a direction field over an appropriate grid for \( \frac{dy}{dx} = f(y) \) (Problem 13) and then for \( \frac{dy}{dx} = f(x) \) (Problem 14).

13. 

**FIGURE 2.1.16** Graph for Problem 13

14. 

**FIGURE 2.1.17** Graph for Problem 14

15. In parts (a) and (b) sketch isoclines \( f(x, y) = c \) (see the Remarks on page 38) for the given differential equation using the indicated values of \( c \). Construct a direction field over a grid by carefully drawing linear elements with the appropriate slope at chosen points on each isocline. In each case, use this rough direction field to sketch an approximate solution curve for the IVP consisting of the DE and the initial condition \( y(0) = 1 \).

(a) \( \frac{dy}{dx} = x + y; c \) an integer satisfying \(-5 \leq c \leq 5\)

(b) \( \frac{dy}{dx} = x^2 + y^2; c = \frac{1}{2}, c = 1, c = \frac{2}{3}, c = 4 \)

16. (a) Consider the direction field of the differential equation \( \frac{dy}{dx} = (y - 4)^2 - 2 \), but do not use technology to obtain it. Describe the slopes of the lineal elements on the lines \( x = 0, y = 3, y = 4 \), and \( y = 5 \).

(b) Consider the IVP \( \frac{dy}{dx} = x(y - 4)^2 - 2, y(0) = y_0 \), where \( y_0 = 4 \). Can a solution \( y(x) \rightarrow \infty \) as \( x \rightarrow \infty \)? Based on the information in part (a), discuss.

17. For a first-order DE \( \frac{dy}{dx} = f(x, y) \) a curve in the plane defined by \( f(x, y) = 0 \) is called a nullcline of the equation, since a lineal element at a point on the curve has zero slope. Use computer software to obtain a direction field over a rectangular grid of points for \( \frac{dy}{dx} = x^2 - 2y \), and then superimpose the graph of the nullcline \( y = \frac{1}{2}x^2 \) over the direction field. Discuss the behavior of solution curves in regions of the plane defined by \( y = \frac{1}{2}x^2 \) and by \( y > \frac{1}{2}x^2 \). Sketch some approximate solution curves. Try to generalize your observations.

18. (a) Identify the nullclines (see Problem 17) in Problems 1, 3, and 4. With a colored pencil, circle any lineal elements in Figures 2.1.12, 2.1.14, and 2.1.15 that you think may be a lineal element at a point on a nullcline.

(b) What are the nullclines of an autonomous first-order DE?

2.1.2 AUTONOMOUS FIRST-ORDER DEs

19. Consider the autonomous first-order differential equation \( \frac{dy}{dx} = y - y^3 \) and the initial condition \( y(0) = y_0 \). By hand, sketch the graph of a typical solution \( y(x) \) when \( y_0 \) has the given values.

(a) \( y_0 > 1 \)

(b) \( 0 < y_0 < 1 \)

(c) \( y_0 < -1 \)

20. Consider the autonomous first-order differential equation \( \frac{dy}{dx} = y^2 - y^4 \) and the initial condition \( y(0) = y_0 \). By hand, sketch the graph of a typical solution \( y(x) \) when \( y_0 \) has the given values.

(a) \( y_0 > 1 \)

(b) \( 0 < y_0 < 1 \)

(c) \( y_0 < -1 \)

In Problems 21–28 find the critical points and phase portrait of the given autonomous first-order differential equation. Classify each critical point as asymptotically stable, unstable, or semi-stable. By hand, sketch typical solution curves in the regions in the \( xy \)-plane determined by the graphs of the equilibrium solutions.

21. \( \frac{dy}{dx} = y^2 - 3y \)

22. \( \frac{dy}{dx} = y^2 - y^3 \)

23. \( \frac{dy}{dx} = (y - 2)^2 \)

24. \( \frac{dy}{dx} = 10 + 3y - y^2 \)

25. \( \frac{dy}{dx} = y^2(4 - y^2) \)

26. \( \frac{dy}{dx} = y(2 - y)(4 - y) \)

27. \( \frac{dy}{dx} = y \ln(y + 2) \)

28. \( \frac{dy}{dx} = ve^{t} - 9y \)

In Problems 29 and 30 consider the autonomous differential equation \( \frac{dy}{dx} = f(y) \), where the graph of \( f \) is given. Use the graph to locate the critical points of each differential
equation. Sketch a phase portrait of each differential equation. By hand, sketch typical solution curves in the subregions in the $xy$-plane determined by the graphs of the equilibrium solutions.

29. Suppose that $\gamma(x)$ is a nonconstant solution of the autonomous equation $dy/dx = f(y)$ and that $c$ is a critical point of the DE. Discuss: Why can’t the graph of $\gamma(x)$ cross the graph of the equilibrium solution $y = c$? Why can’t $f(y)$ change signs in one of the subregions discussed on page 39? Why can’t $\gamma(x)$ be oscillatory or have a relative extremum (maximum or minimum)?

30. Suppose the model in Problem 40 is modified so that air resistance is proportional to $v^2$, that is,

$$
\frac{dv}{dt} = mg - kv^2.
$$

See Problem 17 in Exercises 1.3. Use a phase portrait to find the terminal velocity of the body. Explain your reasoning.

31. Consider the autonomous DE $dy/dx = (2/\pi)y - \sin y$. Determine the critical points of the equation. Discuss a way of obtaining a phase portrait of the equation. Classify the critical points as asymptotically stable, unstable, or semi-stable.

32. A critical point $c$ of an autonomous first-order DE is said to be isolated if there exists some open interval that contains $c$ but no other critical point. Can there exist an autonomous DE of the form given in (2) for which every critical point is nonisolated? Discuss; do not think profound thoughts.

33. Suppose that $\gamma(x)$ is a solution of the autonomous equation $dy/dx = f(y)$ and is bounded above and below by two consecutive critical points $c_1$ and $c_2$, as in subregion $R_3$ of Figure 2.1.6(b). If $f(y) > 0$ in the region, then

$$
\lim_{x \to c_2} \gamma(x) = c_1. \quad \text{Discuss why there cannot exist a number} \quad L \quad \text{such that} \quad \lim_{x \to L} \gamma(x) = L. \quad \text{As part of your} \quad \text{discussion, consider what happens to} \quad \gamma(x) \quad \text{as} \quad x \to \infty.
$$

35. Using the autonomous equation (2), discuss how it is possible to obtain information about the location of points of inflection of a solution curve.

36. Consider the autonomous DE $dy/dx = y^2 - y - 6$. Use your ideas from Problem 35 to find intervals on the $y$-axis for which solution curves are concave up and intervals for which solution curves are concave down. Discuss why each solution curve of an initial-value problem of the form $dy/dx = y^2 - y - 6$, $y(0) = y_0$, where $-2 < y_0 < 3$, has a point of inflection with the same $y$-coordinate. What is that $y$-coordinate? Carefully sketch the solution curve for which $y(0) = -1$. Repeat for $y(2) = 2$.

37. Suppose the autonomous DE in (2) has no critical points. Discuss the behavior of the solutions.

### Mathematical Models

38. **Population Model** The differential equation in Example 3 is a well-known population model. Suppose the DE is changed to

$$
\frac{dP}{dt} = P(aP - b),
$$

where $a$ and $b$ are positive constants. Discuss what happens to the population $P$ as time $t$ increases.

39. **Population Model** Another population model is given by

$$
\frac{dP}{dt} = kP - h,
$$

where $h$ and $k$ are positive constants. For what initial values $P(0) = P_0$ does this model predict that the population will go extinct?

40. **Terminal Velocity** In Section 1.3 we saw that the autonomous differential equation

$$
mg \frac{dv}{dt} = mg - kv,
$$

where $k$ is a positive constant and $g$ is the acceleration due to gravity, is a model for the velocity $v$ of a body of mass $m$ that is falling under the influence of gravity. Because the term $-kv$ represents air resistance, the velocity of a body falling from a great height does not increase without bound as time $t$ increases. Use a phase portrait of the differential equation to find the limiting, or terminal, velocity of the body. Explain your reasoning.