The topological dimension of limits of vertex replacements

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Abstract

Given an initial graph $G$, one may apply a rule $R$ to $G$ which replaces certain vertices of $G$ with other graphs called replacement graphs to obtain a new graph $R(G)$. By iterating this procedure, a sequence of graphs $\{R^n(G)\}$ is obtained. When each graph in this sequence is normalized to have diameter one, the resulting sequence may converge in the Gromov–Hausdorff metric. In this paper, we compute the topological dimension of limit spaces of normalized sequences of iterated vertex replacements involving more than one replacement graph. We also give examples of vertex replacement rules that yield fractals.

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1. Introduction

The notion of a vertex replacement rule was motivated by studying geodesic flows on two-dimensional singular spaces of nonpositive curvature (see [1]). The work in this paper is also related to a class of iterative systems, introduced by Aristid Lindenmayer (see [10, 11]), which is used to model the growth of plants and simple multicellular organisms.
Lindenmayer systems were later used in the areas of data and image compression. Since vertex replacement rules are more natural and geometric, they promise applications in the same fields that Lindenmayer impacted.

A vertex replacement rule $\mathcal{R}$ is a rule for substituting copies of finite graphs (called replacement graphs) for certain vertices in a given graph $G$. The result is a new graph $\mathcal{R}(G)$. Iterating $\mathcal{R}$ produces a sequence of graphs $\{\mathcal{R}^n(G)\}$. By letting $(\mathcal{R}^n(G), 1)$ be the metric space $\mathcal{R}^n(G)$ normalized to have diameter 1, the sequence of the normalized graphs can be studied.

The convergence of the sequence $\{(\mathcal{R}^n(G), 1)\}$ was examined in [5,6]. In [5,7,8] formulas for the topological, box, and Hausdorff dimensions were given for the limit spaces of these sequences when $\mathcal{R}$ is given by only one replacement graph. For the case when $\mathcal{R}$ is given by more than one replacement graph, the formulas for the box and Hausdorff dimensions were computed in [8]. In this paper, we examine the topological dimension of limits of vertex replacements when $\mathcal{R}$ is given by more than one replacement graph. The key result (Theorem 4.2) shows that these limits have topological dimension one. Since a fractal is a metric space with Hausdorff dimension strictly greater than its topological dimension, in the final section we provide examples of fractals which are the limits of vertex replacements involving more than one replacement graph.

2. Vertex replacement rules

In this section we define and provide some basic examples of vertex replacements. Throughout this paper we will assume that all graphs are connected, finite, unit metric graphs, i.e., each graph is a metric space and every edge has length one. In particular, the distance between two points in a graph will be measured by the shortest path in the graph between the two points. For example, in the graph shown in Fig. 1, the distance between vertices $a$ and $b$ is 7.

**Definition 2.1.** A graph $H$ with a designated set of vertices $\{v_1, \ldots, v_k\}$ is called symmetric about $\{v_1, \ldots, v_k\}$ if every permutation of $\{v_1, \ldots, v_k\}$ can be realized by an isometry of $H$.

The vertices in such a designated set are called boundary vertices of $H$ and are denoted by $\partial H$.

**Definition 2.2.** A vertex replacement rule $\mathcal{R}$ consists of a finite list of finite graphs (called replacement graphs) $\{H_1, \ldots, H_p\}$, each with a set $\partial H_i$ of boundary vertices, so that $|\partial H_i| \neq |\partial H_j|$ for $i \neq j$, where $|\cdot|$ denotes the cardinality of a set.

![Fig. 1.](image)
Let $G$ be a graph and let $\mathcal{R}$ be a vertex replacement rule given by the replacement graphs $H_1, \ldots, H_p$. Recall that the degree of a vertex $v$ in $G$, denoted $\deg(v)$, is the number of edges in $G$ adjacent to $v$.

**Definition 2.3.** A vertex $v$ in $G$ is called replaceable if $\deg(v) = |\partial H_i|$ for some replacement graph $H_i$ in the replacement rule.

The replacement rule $\mathcal{R}$ acts on $G$ by substituting each replaceable vertex in $G$ with its corresponding replacement graph so that the $\deg(v)$ edges previously attached to $v$ in $G$ are attached to the $|\partial H_i|$ vertices of $H_i$. Since $|\partial H_i| \neq |\partial H_j|$ for $i \neq j$, each replaceable vertex has a unique corresponding replacement graph. Also, since each replacement graph $H_i$ is symmetric about $\partial H_i$, it is irrelevant how the edges previously adjacent to $v$ are attached to $H_i$. Thus, vertex replacement is a well defined procedure.

For example, we may define a vertex replacement rule $\mathcal{R}$ by the replacement graphs $H_1$ and $H_2$ depicted in Fig. 2. The boundary vertices of the replacement graphs are shown with circles. Note that each replacement graph is symmetric about its set of boundary vertices. Let $G$ be as depicted in Fig. 3. Vertices $w_1$, $w_2$, and $w_3$ are replaceable by $H_1$, and vertices $v_1$, $v_2$, and $v_3$ are replaceable by $H_2$, but vertices $x_1$, $x_2$, and $x_3$ are not replaceable. Fig. 4 shows $\mathcal{R}(G)$.

We extend the idea of a replaceable vertex to include the vertices of the replacement graphs themselves, but only after the replacement graphs have replaced some vertices. That is, one should not treat a replacement graph $H_i$ as an initial graph $G$, but always view it as having already replaced some vertex. Hence we view each boundary vertex as having another edge attached.
Definition 2.4. A boundary vertex $v \in \partial H_j$ is called replaceable if $\deg(v) = |\partial H_j| - 1$ for some replacement graph $H_j$ in the replacement rule.

Notice that for the replacement rule in Fig. 2, the boundary vertices of $H_1$ are replaceable by $H_2$ (each such vertex will have 3 edges adjacent after being inserted into a graph $G$) while the remaining vertices of $H_1$ are replaceable by $H_1$. Likewise, the boundary vertices of $H_2$ are replaceable by $H_2$ while the remaining vertices of $H_2$ are replaceable by $H_1$. Thus the replacement rule $\mathcal{R}$ may be iterated to create a sequence of graphs $\{\mathcal{R}^n(G)\}$.

When each graph in this sequence is scaled to have diameter one, we obtain the sequence $\{(\mathcal{R}^n(G), 1)\}$ which, according to Theorem 3.3, will converge in the Gromov–Hausdorff metric. Fig. 5 shows the next two graphs in the sequence and the limit space of this sequence.

**Fig. 4.** The graph $\mathcal{R}(G)$.

**Fig. 5.** $\mathcal{R}^2(G), 1)$, $\mathcal{R}^3(G), 1)$, and the limit of $\{(\mathcal{R}^n(G), 1)\}$. 
In general, for any set $F$ in $G$, let $R(F)$ be $\pi^{-1}(F)$. If $F \subseteq G$ contains no replaceable vertices, then $R^n(F)$ can be identified with $F$ and we label $R^n(F)$ as $F \subseteq R^n(G)$. Similarly, if $z \in G$ is not replaceable, label $R^n(z)$ as $z \in R^n(G)$. Observe that the inverse map $\pi$ is only well defined when one is also given a specific replacement rule $\mathcal{R} : G \rightarrow R(G)$. Otherwise, given a graph $F$, there might be two different graphs $G_1$ and $G_2$ such that $R(G_1) = F = R(G_2)$, and thus, two different inverse maps $\pi_1 : F \rightarrow G_1$ and $\pi_2 : F \rightarrow G_2$.

Let $F$ be any finite graph and let $\gamma$ be a simple path in $F$. Let $L(\gamma)$ denote the length of $\gamma$. Let $N_{i}(F)$ denote the number of vertices in $F$ which are replaceable by $H_i$, and let $N(F)$ be the total number of replaceable vertices in $F$. That is, for a replacement rule with $p$ replacement graphs, $N(F) = \sum_{i=1}^{p} N_{i}(F)$. For a replacement graph $H_i$, we define $N_{i}(H_i)$ to be the number of vertices in $H_i$ replaceable by $H_j$ when one regards $H_i$ as a subset of $R(G)$. That is, $N_{i}(H_j)$ is the number of vertices $v$ in $H_j$ such that $\deg(v) = |\partial H_j| - 1$ if $v$ is a boundary vertex or $\deg(v) = |\partial H_j|$ if $v$ is not a boundary vertex.

Let $H_i$ be a replacement graph in a replacement rule $\mathcal{R}$ and let $v_i$ be a vertex in a graph $G$ which is replaceable by $H_i$. Define the set $\partial R^n(v_i)$ to be all vertices $w \in R^n(v_i)$ that are adjacent to one of the $\deg(v_i)$ edges outside of $R^n(v_i)$ that were adjacent to $v_i \in G$. So $\partial R^n(v_i)$ is the set of possible vertices through which a path in $R^n(G)$ must pass when entering or exiting $R^n(v_i) \subseteq R^n(G)$. Note that $|\partial R^n(v_i)| = |\partial H_i|$. For example, if $R$ is the replacement rule given in Fig. 2 and $w_3$ is as in Fig. 3, then Fig. 6 depicts the two vertices in $\partial R^3(w_3)$ with circles. To determine the growth of $\text{diam}(R^n(G))$ (and the growth in complexity of $(R^n(G), 1)$), we need to measure the distance between points in $\partial R^n(v_i)$.

Hence we define the function

$$a_i(n) = \text{dist}_{R^n(v_i)}(u, u'),$$

where $u, u' \in \partial R^n(v_i)$ for $u \neq u'$. We also define the function

$$b_i(n) = \sup_{v \in \partial R^n(v_i)} \text{dist}_{R^n(v_i)}(u, v) \mid u \in \partial R^n(v_i).$$

By the symmetry of each $H_i$ about $\partial H_i$, the above definitions are independent of the choices of $u$ and $u'$ in $\partial R^n(v_i)$. Clearly $a_i(n) \leq b_i(n)$. Let $a_{\text{max}}(n) = \max_{i=1,\ldots,p} a_i(n)$, $a_{\text{min}}(n) = \min_{i=1,\ldots,p} a_i(n)$, $b_{\text{max}}(n) = \max_{i=1,\ldots,p} b_i(n)$, and $b_{\text{min}}(n) = \min_{i=1,\ldots,p} b_i(n)$.
Definition 2.5. A path $\sigma$ in a replacement graph $H_i$ is called a simple boundary connecting path if $\sigma$ is a simple path with boundary vertices for endpoints and no boundary vertices on its interior.

One example of a simple boundary connecting path is the ‘trivial path’ from a boundary vertex to itself which consists of only the boundary vertex.

For each $n$, there is a path in $R^n(v_i)$ that realizes $a_i(n)$ and projects via $\pi^{n-1}$ to a simple boundary connecting path $\sigma_i(n)$ in $H_i$. It is extremely difficult to combinatorially determine the $\sigma_i(n)$ given an arbitrary replacement rule $R$ since, in general, $\sigma_i(n) \neq \sigma_i(m)$ for $n \neq m$. Hence, we restrict to simple replacement rules.

Definition 2.6. A replacement rule $R$ given by the graphs $H_1, \ldots, H_p$ is simple if there is a unique matrix $\tilde{A}$ such that for any set of simple boundary connecting paths $\{\sigma_1, \ldots, \sigma_p\}$, where $\sigma_i \subset H_i$, we have

$$\tilde{A} = \begin{bmatrix} A & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} N_1(\sigma_1) & \cdots & N_p(\sigma_1) & L(\sigma_1) \\ \vdots & \ddots & \vdots & \vdots \\ N_1(\sigma_p) & \cdots & N_p(\sigma_p) & L(\sigma_p) \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Let $A$ denote the upper left $p \times p$ block of $\tilde{A}$. We call the matrix $A$ a path matrix of $R$.

Definition 2.7. If every path matrix $A$ of a replacement rule $R$ is primitive (i.e., $A^k$ has only positive entries for some power $k$), then $R$ is called primitive.

In the case where a replacement graph $H_i$ has only one boundary vertex (and hence it has no path between distinct boundary vertices), then $L(\sigma_i) = 0$ and the row $N_1(\sigma_i), \ldots, N_p(\sigma_i)$ in the path matrix $A$ above is either a row of zeros (when the boundary vertex of $H_i$ is nonreplaceable) or else a row in which all but the $j$th entry is a zero (when the boundary vertex of $H_i$ is replaceable by $H_j$).

The replacement rule in Fig. 2 is simple and primitive. So if $\sigma_1$ and $\sigma_2$ are simple boundary connecting paths in $H_1$ and $H_2$, respectively, then

$$\tilde{A} = \begin{bmatrix} N_1(\sigma_1) & N_2(\sigma_1) & L(\sigma_1) \\ N_1(\sigma_2) & N_2(\sigma_2) & L(\sigma_2) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Of course not every replacement rule is simple and primitive. Fig. 7 shows an example of a replacement rule which is simple but not primitive since its path matrix is $\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. The replacement rule in Fig. 8 is not simple because there are two simple boundary connecting paths in $H_1$ which give rise to two different path matrices: $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ comes from taking the upper route between the boundary vertices of $H_1$ and $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$ comes from taking the lower route between the boundary vertices of $H_1$. 
3. Convergence results

Before stating the convergence results, let us recall some facts about the metric in which these sequences converge—the Gromov–Hausdorff metric. For any metric space $X$, $\text{dist}_X$ will denote the metric on $X$. Let $Z$ be a metric space. For $C \subset Z$ and $\varepsilon > 0$, let $C_\varepsilon = \{z \in Z : \text{dist}_Z(z, C) < \varepsilon\}$.

**Definition 3.1.** The **Hausdorff distance** between two nonempty compact subsets $A$ and $B$ of a metric space $Z$ is defined by

$$\text{dist}_{Z}^{\text{Haus}}(A, B) = \inf\{\varepsilon > 0 : A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\}.$$  

The Hausdorff distance defines a metric on the set of all compact subsets of $Z$.

We are now able to define the Gromov–Hausdorff distance. Informally, when measuring the Gromov–Hausdorff distance between spaces $X$ and $X'$, we place $X$ and $X'$ into some space in such a way that they are as close together as possible and then measure the resulting Hausdorff distance. Let $\mathcal{S}$ denote the collection of all isometry classes of compact metric spaces.

**Definition 3.2.** The **Gromov–Hausdorff distance** between two compact metric spaces $X$ and $X'$ is defined by

$$\text{dist}_{Z}^{\text{GH}}(X, X') = \inf_{I, J} \{\varepsilon > 0 : \text{dist}_{Z}^{\text{Haus}}(I(X), J(X')) < \varepsilon\},$$

where $I$ and $J$ are isometric embeddings of $X$ and $X'$ into $Z$, respectively.
The space \((S, \text{dist}_{GH}^S)\) is a complete metric space. Moreover, \(\text{dist}_{GH}^S(X, X') = 0\) if and only if \(X\) is isometric to \(X'\). (See [4].)

For a finite graph \(G\), let \((\mathcal{R}^n(G), 1)\) be the metric space \(\mathcal{R}^n(G)\) normalized to have diameter 1, i.e., every edge in \((\mathcal{R}^n(G), 1)\) has length \(1/\text{diam}(\mathcal{R}^n(G))\).

**Theorem 3.3** (J. Previte, M. Previte and M. Vandescos). Let \(H_1, \ldots, H_p\) define a simple, primitive vertex replacement rule \(R\) with \(p \geq 2\) and let \(G\) be a finite graph with at least one replaceable vertex. Then the normalized sequence \(\{(\mathcal{R}^n(G), 1)\}\) converges in the Gromov–Hausdorff metric.

It should be noted that Theorem 3.3 extends to primitive replacement rules which are eventually simple (see [6]), but to streamline the arguments, this paper will discuss only simple replacement rules.

We conclude this section with two lemmas required to prove the formulas for the dimensions of limits of vertex replacements. The first lemma follows from Perron–Frobenius theory. See [2, p. 45] and [6, Corollary 3.9] for details.

**Lemma 3.4.** Let \(A\) be a nonnegative primitive \(p \times p\) matrix with integer entries and \(p \geq 2\). Then the spectral radius of \(A\) is greater than 1.

The next lemma is a result from [6]. It says that the \(a_i\)’s and \(b_i\)’s grow at the same rate.

**Lemma 3.5.** If the replacement rule \(R\) is simple and primitive and \(\rho\) is the spectral radius of the path matrix of \(R\), then there exist positive constants \(\tilde{K}\), \(\kappa_1\), and \(\kappa_2\) such that

\[
\kappa_1 \leq \frac{a_i(n)}{b_j(n)} \leq \kappa_2, \tag{1}
\]

\[
\frac{\kappa_1}{\rho^m} \leq \frac{b_i(n)}{b_j(n + m)} \leq \frac{\kappa_2}{\rho^m}, \tag{2}
\]

and

\[
\frac{\kappa_1}{\rho^m} \leq \frac{a_i(n)}{b_j(n + m)} \leq \frac{\kappa_2}{\rho^m} \tag{3}
\]

for all \(i, j = 1, \ldots, p\) and for all \(n > \tilde{K}\).

**4. The topological dimension of limits of vertex replacements**

Recall the topological dimension of a topological space \(X\). See [9, Section 50].

**Definition 4.1.** A space \(X\) is said to have topological dimension \(m\) if \(m\) is the smallest integer such that for every open covering \(\mathcal{A}\) of \(X\), there is an open covering \(\mathcal{B}\) of \(X\) that refines \(\mathcal{A}\) and has order at most \(m + 1\).

We now state and prove our main result.
Theorem 4.2. Let $G$ be a finite graph and let $H_1, \ldots, H_p$ define a simple, primitive vertex replacement rule $R$. Then the sequence $\{(R^n(G), 1)\}$ converges in the Gromov–Hausdorff metric to a metric space $X$ which has topological dimension 1.

Proof. If $G$ contains no replaceable vertex, then the result clearly holds. Therefore, we assume that $G$ contains at least one replaceable vertex. Let $A$ be an open cover of $X$. Since $X$ is compact, $A$ has a Lebesgue number $\delta > 0$. We now construct an order 2 refinement $B$ of $A$ which will consist of two types of open sets. The first type essentially covers the limits of replaceable vertices in $R^m(G)$, where $m$ is sufficiently large. The second type consists of open balls centered at special points in $X$. We begin with the first type.

Let $V$ denote the set of replaceable vertices in $R^m(G)$. As $n \to \infty$ the sequence $\{(R^{n+m}(G), 1)\}$ of graphs converges to $X$ in the Gromov–Hausdorff metric. Therefore, for each $v$ in $V \subset R^m(G)$, there is a subset $Y_v \subset X$ such that $\{(R^n(v), 1)\}$ converges to $Y_v$ in the Gromov–Hausdorff metric.

Claim 1. For sufficiently large $m$, the diameter of $Y_v$ is less than the Lebesgue number $\delta$ for all $v \in V \subset R^m(G)$.

Proof. Since $Y_v = \lim_{n \to \infty}(R^n(v), 1)$, then

$$\text{diam}(Y_v) = \lim_{n \to \infty} \frac{\text{diam}(R^n(v))}{\text{diam}(R^{n+m}(G))}.$$ 

Any path $\gamma$ that realizes diam$(R^n(v))$ projects via $\pi^{n-1}$ to a path $\gamma'$ in a replacement graph which passes through at most max $(N(H_i))$ replaceable vertices. Let $w \in \gamma'$ be one such replaceable vertex. Since $\gamma$ passes through $R^{n-1}(w)$, then the length of $\gamma \cap R^{n-1}(w)$ is at most $d_{\text{max}}(n - 1)$ if $w$ is an interior vertex of $\gamma'$ and it is at most $b_{\text{max}}(n - 1)$ if $w$ is an endpoint of $\gamma'$. Furthermore, since $\gamma$ is a distance minimizing path, then its projection $\gamma'$ has length at most $\text{diam}(H_i)$. Thus,

$$\text{diam}(R^n(v)) \leq 2b_{\text{max}}(n - 1) + \max_i N(H_i) - 2d_{\text{max}}(n - 1) + 2\text{diam}(H_i).$$

Furthermore, since $G$ contains at least one replaceable vertex, then diam$(R^{n+m}(G)) \geq b_{\text{min}}(n + m - 1)$. Therefore, by Lemma 3.5 there is a positive constant $k$ which depends only upon $R$ such that

$$0 < \text{diam}(R^n(v)) \leq \lim_{n \to \infty} \frac{2b_{\text{max}}(n - 1) + \max_i N(H_i) - 2d_{\text{max}}(n - 1) + \max_i \text{diam}(H_i)}{b_{\text{min}}(n + m - 1)} \leq \frac{k}{\rho^m}.$$ 

By Lemma 3.4, the spectral radius $\rho$ of the path matrix of $R$ is greater than 1. Thus, for $m$ large enough, diam$(Y_v) < \delta/2$ for all replaceable vertices $v \in V \subset R^m(G)$.

Let $Y_v^0$ denote the interior of $Y_v$. Note that for distinct replaceable vertices $v$ and $w$ in $V$, the sets $Y_v^0$ and $Y_w^0$ are pairwise disjoint in $X$. These open sets form the first part of our
Let $P := X \setminus \bigcup_{v \in V} Y_v^\circ$.

Claim 2. $P$ is a finite set.

Proof. Let $x \in \mathcal{R}^m(G) \setminus V$. Since $V$ is the set of all replaceable vertices in $\mathcal{R}^m(G)$, then the distance between $x$ and $V$ is realized by a nonreplaceable path $y$ in the connected component of $\mathcal{R}^m(G) \setminus V$ which contains $x$. Thus, $\text{dist}_{\mathcal{R}^m(G)}(\mathcal{R}^m(V), x)$ is the length of $y$ for all $n > 0$. Hence, $\lim_{n \to \infty} \text{dist}_{\mathcal{R}^m(G)}(\mathcal{R}^m(V), x) = 0$. In particular, if $x$ and $y$ belong to the same connected component of $\mathcal{R}^m(G) \setminus V$, then $\lim_{n \to \infty} \text{dist}_{\mathcal{R}^m(G)}(\mathcal{R}^m(V), x, y) = 0$.

However, if $x$ and $y$ belong to distinct connected components of $\mathcal{R}^m(G) \setminus V$, then any distance realizing path in $\mathcal{R}^m(G)$ between $x$ and $y$ must pass through a replaceable vertex. So $\text{dist}_{\mathcal{R}^m(G)}(x, y) \geq a_{\min}(n)$ for all $n > 0$. Since $\text{diam}(\mathcal{R}^{m+1}(G)) \leq \text{diam}(\mathcal{R}^m(G)) + |V|b_{\max}(n)$, then

$$
\lim_{n \to \infty} \text{dist}_{\mathcal{R}^{m+1}(G)}(x, y) \geq \lim_{n \to \infty} \frac{a_{\min}(n)}{\text{diam}(\mathcal{R}^m(G)) + |V|b_{\max}(n)}.
$$

Since $m$ is fixed and $n \to \infty$, then we may assume $\text{diam}(\mathcal{R}^m(G)) \leq |V|b_{\max}(n)$. Hence, by Lemma 3.5 we have

$$
\lim_{n \to \infty} \text{dist}_{\mathcal{R}^{m+1}(G)}(x, y) \geq \lim_{n \to \infty} \frac{a_{\min}(n)}{2|V|b_{\max}(n)} \geq \frac{k_1}{2|V|} > 0,
$$

where $k_1$ is a positive constant from Lemma 3.5. Therefore, $|P|$ is the number of connected components of $\mathcal{R}^m(G) \setminus V$. Thus, $P$ is discrete. Hence, it is finite since $X$ is compact. \hfill \Box

Without loss of generality, we may assume $|P| > 1$. Let $d = \inf\{\text{dist}_X(p, q) : p, q \in P, p \neq q\}$. Since $P$ is a finite set, then $d > 0$. The second part of our refinement $B$ is the set of open balls $B_p$ in $X$ centered at $p \in P$ with radius $\frac{1}{2} \min(\delta, d)$. For distinct points $p$ and $q$ in $P$, we have that $B_p \cap B_q = \emptyset$, and the diameter of each ball is less than the Lebesgue number $\delta$ of the open cover $A$ of $X$.

Since for any $x \in \mathcal{R}^m(G) \setminus V$, we have $\lim_{n \to \infty} \text{dist}_{\mathcal{R}^{m+1}(G)}(\mathcal{R}^m(V), x) = 0$, then $\{Y_v^\circ|v \in V\}$ forms a cover of $X$. Since $\{Y_v^\circ|v \in V\}$ forms a cover of $X$ and $Y_v^\circ = Y_v^\circ \setminus P$, then the collection $B$ consisting the open sets in $\{Y_v^\circ|v \in V\}$ together with the open balls in the set $\{B_p|p \in P\}$ forms an open cover of $X$. Also, no point of $X$ is contained in any three elements of $B$ since the $Y_v^\circ$'s are pairwise disjoint and the open balls are pairwise disjoint. Finally, the diameter of every element in $B$ is less than $\delta$. Thus, $B$ is an order 2 refinement of $A$ and the topological dimension of $X$ is 1. \hfill \Box

5. Examples

Recall that a fractal is a metric space with Hausdorff dimension strictly greater than its topological dimension. (See the introduction of [3].) Using the formula given in [8] for the Hausdorff dimension of limits of vertex replacement rules, one can easily create
replacement rules that yield metric spaces with Hausdorff dimension greater than one. The examples below illustrate a few such replacement rules. As in earlier examples, the boundary vertices of the replacement graphs are depicted using circles.

**Example 1.** For our first example, the replacement rule $R$ and initial graph $G$ are shown in Fig. 9, and a few iterations of the replacement are shown in Fig. 10. The sequence $\{(R^n(G), 1)\}$ converges to the metric space having topological dimension 1 and Hausdorff dimension $\frac{\ln(5+\sqrt{57})}{\ln 2}$, which is depicted in Fig. 11.

**Example 2.** We now give an example which must be embedded in at least three dimensions (Fig. 12). A few iterations of the replacement are shown in Fig. 13, and the limit space of the sequence $\{(R^n(G), 1)\}$ is depicted in Fig. 14. Its Hausdorff dimension is $\frac{\ln(3+\sqrt{7})}{\ln 3}$.

Fig. 9. A replacement rule $R = \{H_1, H_2\}$ and an initial graph $G$.

Fig. 10. $(R(G), 1)$, $(R^2(G), 1)$, and $(R^3(G), 1)$.

Fig. 11. A doily.
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Example 3. In our final example, the replacement rule \( \mathcal{R} \) (Fig. 15) has path matrix \( A = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \). Note that although \( A \) is not positive, it is primitive since \( A^2 \) is positive. Consequently, \( \mathcal{R} \) is a primitive replacement rule. Therefore, for any initial graph \( G \) with at least one replaceable vertex, the limit of the sequence \( \{(\mathcal{R}^n(G), 1)\} \) has topological dimension 1.
and Hausdorff dimension $\frac{\ln 6}{\ln(1+\sqrt{3})}$. Fig. 16 shows a few iterations of the replacement for the initial graph $G$ given in Fig. 15.

**References**


