The topological dimension of limits of graph substitutions

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Abstract. In 1998, J. Previte developed a framework for studying the dynamics of iterated replacements of certain vertices in a finite graph \(G\) by a finite graph \(H\) (see [3]). He showed that, except for special cases, the sequence of graphs formed by iterating vertex replacements converges in the Gromov-Hausdorff metric. In this paper we prove that the topological dimension of these limit spaces is one. We also provide examples of graph substitutions whose limit spaces are fractals.

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1 Introduction

A vertex replacement rule \(\mathcal{R}\) is a rule for substituting copies of finite graphs for certain vertices in a given graph \(G\). The result is a new graph \(\mathcal{R}(G)\). A sequence of graphs \(\mathcal{R}^n(G)\) is produced by iterating \(\mathcal{R}\).

In [3], J. Previte found necessary and sufficient conditions for normalized iterations of a replacement rule with one substitution graph to converge in the Gromov-Hausdorff metric. He also determined the Hausdorff dimension of the limit spaces. Our key result about these limit spaces is that they have topological dimension one.

For our purposes, a fractal is defined as a metric space with topological dimension strictly less than its Hausdorff dimension. Many standard examples of fractals, such as the Sierpinski triangle and the Sierpinski tetrahedron, are limit spaces of normalized sequences of graph substitutions. In the final section of this paper we construct several examples of graph substitutions whose limit spaces are fractals.

2 Graph substitutions

In this section we define and provide some basic examples of graph substitutions. Throughout this paper we will assume that all graphs are connected, locally finite, unit metric graphs, i.e., each graph is a metric space and every edge has length one. For any graph \(G\), let \(V(G)\) be the set of vertices of \(G\).

A graph \(H\) with a selected set of vertices \(\{v_1, \ldots, v_k\} \subset V(H)\) is called symmetric about \(\{v_1, \ldots, v_k\}\) if every permutation of \(\{v_1, \ldots, v_k\}\) can be realized by an isometry
of $H$. Such vertices are called boundary vertices. A vertex replacement rule $\mathcal{R}$ is a rule for substituting copies of finite graphs for certain vertices in a given graph $G$. For the purpose of this paper, a vertex replacement rule $\mathcal{R}$ is completely determined by a finite graph $H$ called a replacement graph with a set of boundary vertices, denoted by $\partial H$.

Let $G$ be a graph and let $\mathcal{R}$ be a vertex replacement rule $H$. The degree of a vertex $v \in V(G)$, denoted by $\deg(v)$, is the number of edges in $G$ incident to $v$. A vertex $v \in V(G)$ is called replaceable if $\deg(v) = |\partial H|$, where $|\cdot|$ denotes the cardinality of a set. The replacement rule $\mathcal{R}$ acts on $G$ by substituting each replaceable vertex in $G$ with a copy of $H$, producing a new graph $\mathcal{R}(G)$. In particular, each vertex $v \in V(G)$ with $\deg(v) = |\partial H|$ is replaced with a copy of the graph $H$ so that the $\deg(v)$ edges previously attached to $v$ in $G$ are attached to $|\partial H|$ vertices of $H$. This procedure is well defined since the graph $H$ is symmetric about $\partial H$.

In all of our examples, we indicate the boundary vertices of each replacement graph by open circles.

**Example 1.** Consider the vertex replacement rule $\mathcal{R}$ given by $H$ of Figure 1.

![Figure 1](image1.png)

*Fig. 1. A replacement rule $\mathcal{R}$*

Let $G$ be as in Figure 2 (a). Observe that $v$ and $w$ are the only replaceable vertices in $G$. The construction of $\mathcal{R}(G)$ is shown below in Figures 2 (b) and (c).

Note that in Example 1, after substituting two copies of $H$ into $G$, the boundary vertices become replaceable in $\mathcal{R}(G)$. In general, we say that a boundary vertex $v \in \partial H$ is replaceable if and only if $\deg(v) = |\partial H| - 1$.

![Figure 2](image2.png)

*Fig. 2. Construction of $\mathcal{R}(G)$*
Example 2. Let \( H \) define the vertex replacement rule \( R \) with boundary vertices as shown in Figure 3 and let \( G \) be the graph given in the figure. Observe that the boundary vertices of \( H \) are replaceable. The graphs \( R(G) \) and \( R^2(G) \) are given in Figure 4.

For a finite graph \( G \), let \( (R^n(G), 1) \) be the metric space \( R^n(G) \) normalized to have diameter 1, i.e., every edge in \( (R^n(G), 1) \) has length \( 1/\text{diam}(R^n(G)) \). In [3], J. Previte proved that for a vertex replacement rule given by a graph \( H \), the normalized sequence of graphs \( (R^n(G), 1) \) converges (except for special cases) in the Gromov-Hausdorff metric. He also calculated the Hausdorff dimension of the limit space for the case when \( H \) has exactly one replaceable vertex and the case when the boundary vertices of \( H \) are replaceable. Our main result is that the topological dimension of these limit spaces is one (Theorem 4.1).

3 Main definitions and convergence results

This section includes our principal definitions and several of J. Previte’s convergence results for sequences of graph substitutions that will be used in the proof of our main result.
Let us recall some facts about the Gromov-Hausdorff metric. For any metric space $X$, $\text{dist}(X)$ will denote the metric on $X$. Let $Z$ be a metric space. For $C \subset Z$ and $\varepsilon > 0$, let $C_\varepsilon = \{z \in Z : \text{dist}(z, C) < \varepsilon\}$. The Hausdorff distance between two nonempty compact subsets $A$ and $B$ of $Z$ is defined by

$$\text{dist}^\text{Haus}(A, B) = \inf \{\varepsilon > 0 : A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\}.$$ 

This defines a metric on the set of all compact subsets of $Z$.

Let $\mathcal{M}$ denote the collection of all isometry classes of compact metric spaces. The Gromov-Hausdorff distance between two compact metric spaces $X$ and $Y$ is defined by

$$\text{dist}^\text{GH}(X, Y) = \inf_{I, J} \{\varepsilon > 0 : \text{dist}^\text{Haus}(I(X), J(Y)) < \varepsilon\},$$

where $I$ and $J$ are isometric embeddings of $X$ and $Y$ into $Z$, respectively. The space $(\mathcal{M}, \text{dist}^\text{GH})$ is a complete metric space. Moreover, $\text{dist}^\text{GH}(X, Y) = 0$ if and only if $X$ is isometric to $Y$.

Let $H$ define a vertex replacement rule $\mathcal{R}$ and let $G$ be a finite graph. For a replaceable vertex $v$ in $G$, let $\mathcal{R}(v)$ be the copy of the graph $H$ in $\mathcal{R}(G)$ that replaced the vertex $v$. There exists a pointwise map $\pi : \mathcal{R}(G) \to G$ defined by $\pi(\mathcal{R}(v)) = v$. The map $\pi$ undoes replacement by crushing the inserted $H$’s to the vertices they replaced. In general, for any set $F$ in $G$, let $\mathcal{R}(F)$ be $\pi^{-1}(F)$. If $F \subset G$ contains no replaceable vertices, then $\mathcal{R}^n(F)$ can be identified with $F$ and we label $\mathcal{R}^n(F)$ as $F \subset \mathcal{R}^n(G)$. Similarly, if $z \in G$ is not replaceable, label $\mathcal{R}^n(z)$ as $z \in \mathcal{R}^n(G)$.

For any path $\eta$ in $G$, let $L(\eta)$ denote the length of $\eta$. A path $\eta \subset G$ is called distance minimizing if $L(\eta) = \text{dist}(x, y)$, where $x$ and $y$ are the endpoints of $\eta$. A path $\zeta$ in $G$ realizes the diameter of $G$ if $\zeta$ is distance minimizing and $L(\zeta) = \text{diam}(G)$.

For any subset $F$ of $G$, let $N(F)$ denote the number of replaceable vertices contained in $F$. For a replacement graph $H$, we define $N(H)$ to be the number of replaceable vertices in $H$ when one regards $H$ as a subset of $\mathcal{R}(G)$.

A path $\eta$ in $G$ is called preminimizing if it is a distance minimizing path between its endpoints that contains the least number of replaceable vertices. That is, if $\eta'$ is any path between the endpoints of $\eta$, then

1. $N(\eta') \geq N(\eta)$, and
2. if $N(\eta') = N(\eta)$, then $L(\eta') \geq L(\eta)$.

For any two distinct boundary vertices $b_1, b_2 \in \partial H$, define $W_H$ to be $N(\sigma)$, where $\sigma$ is a preminimizing path from $b_1$ to $b_2$. By symmetry of $H$ about $\partial H$, if $\sigma'$ is a preminimizing path between any other pair of distinct boundary vertices $b_1'$ and $b_2'$, then $N(\sigma) = N(\sigma')$. Thus the number $W_H$ is well defined.

We are now able to state when the sequence of replacement graphs $(\mathcal{R}^n(G), 1)$ converges in the Gromov-Hausdorff metric.

**Theorem 3.1** (J. Previte). Let $H$ define a replacement rule $\mathcal{R}$ and let $G$ be a finite graph...
with at least one replaceable vertex. Then the normalized sequence \((\mathbb{R}^n(G), 1)\) converges in the Gromov-Hausdorff metric if and only if one of the following hold:

1. \(N(H) = 1\), or
2. \(|\partial H| \geq 2\) and \(W_H \geq 2\).

In the case where \(N(H) = 1\), J. Previte showed that the limit space \(X\) has topological and Hausdorff dimension 1. In the second case, he showed that the Hausdorff dimension of \(X\) is \(\dim_H(X) = \frac{\ln N(H)}{\ln W_H}\), provided that the boundary vertices of \(H\) are replaceable. Henceforth, we will assume that \(|\partial H| \geq 2\) and \(W_H \geq 2\).

Preminimizing paths are important tools for finding bounds on the lengths of distance minimizing paths. The following lemma shows the relationship between distance minimizing and preminimizing paths.

**Lemma 3.2 (J. Previte).** Let \(G\) be a finite graph. There exists a number \(k(G)\) so that if \(x\) is a distance minimizing path in \(\mathbb{R}^n(G)\) and \(n > k(G)\), then \(p_n(x)\) is preminimizing in \(G\).

Define the set \(\partial \mathbb{R}^n(v)\) to be all vertices in \(\mathbb{R}^n(v)\) that are adjacent to one of the \(|\partial H|\) edges outside of \(\mathbb{R}^n(v)\). Note that \(|\partial \mathbb{R}^n(v)| = |\partial H|\). The following functions will be used to calculate the lengths of paths:

\[
a(n) = \text{dist}_{\mathbb{R}^n(v)}(u, u') \quad \text{where } u, u' \in \partial \mathbb{R}^n(v) \text{ for } u \neq u'
\]

and

\[
b(n) = \sup_{z \in \mathbb{R}^n(v)} \{\text{dist}_{\mathbb{R}^n(v)}(u, z) : u \in \partial \mathbb{R}^n(v)\}.
\]

Note that \(a(n) \leq b(n)\). Again, by symmetry, these functions are independent of the choices of \(u, u' \in \partial \mathbb{R}^n(v)\).

For any boundary vertex \(u \in \partial H\), let \(M_H\) be the maximum of \(N(\eta)\) where \(\eta \subset H\) is a preminimizing path with \(u\) as one endpoint. By symmetry, \(M_H\) is independent of \(u\). The following result will be used in the proof of the main theorem.

**Lemma 3.3 (J. Previte).** If \(W_H \geq 2\), then

\[
\lim_{n \to \infty} \frac{a(n)}{b(n)} = \frac{W_H - 1}{M_H - 1}.
\]

We end this section by recalling some facts about the topological dimension of a space \(X\). A collection \(\mathcal{A}\) of subsets of a topological space \(X\) is said to have order \(m + 1\) if some point of \(X\) lies in \(m + 1\) elements of \(\mathcal{A}\), and no point of \(X\) lies in more than \(m + 1\) elements of \(\mathcal{A}\). A space \(X\) is said to have topological dimension \(m\) if \(m\) is the smallest integer such that for every open covering \(\mathcal{A}\) of \(X\), there is an open covering \(\mathcal{B}\) of \(X\) that refines \(\mathcal{A}\) and has order at most \(m + 1\). If \(\mathcal{A}\) is an open covering of a compact metric space \(X\), then there is a number \(\delta > 0\), called a *Lebesgue number*,
such that for each subset of $X$ having diameter less than $\delta$, there exists an element of $\mathcal{A}$ containing it.

4 The topological dimension of limit spaces

Let $G$ be a finite graph and let $H$ define a replacement rule $\mathcal{R}$. Recall that if $|\partial H| \geq 2$, $W_H \geq 2$, and the boundary vertices of $H$ are replaceable, then the Hausdorff dimension of the limit space $X$ of the sequence $(\mathcal{R}^n(G), 1)$ is given by $\dim_H(X) = \frac{\ln N(H)}{\ln W_H}$. We now show that the topological dimension of $X$ is 1, thereby giving fractals when $N(H) > W_H$.

Theorem 4.1. Let $G$ be a finite graph and let $H$ define a vertex replacement rule $\mathcal{R}$. Suppose the sequence $(\mathcal{R}^n(G), 1)$ converges in the Gromov-Hausdorff metric to a metric space $X$. If $|\partial H| \geq 2$ and $W_H \geq 2$, then the topological dimension of $X$ is 1.

Proof. Let $\mathcal{A}$ be an open cover of $X$. Since $X$ is compact, $\mathcal{A}$ has a Lesbegue number $\delta > 0$. We now construct an order 2 refinement $\mathcal{B}$ of $\mathcal{A}$ which will consist of two types of open sets. The first type essentially covers the limits of certain replaceable vertices. The second type consists of open balls centered at special points in $X$. We begin with the first type.

Let $V$ denote the set of replaceable vertices in $\mathcal{R}^{m-1}(G)$. Since as $n \to \infty$, the sequence of graphs $(\mathcal{R}^{n+m-1}(G), 1)$ converges to $X$ in the Gromov-Hausdorff metric, then for each $v \in V$, there is a subset $Y_v \subset X$ such that $(\mathcal{R}^n(v), 1)$ converges to $Y_v$ in the Gromov-Hausdorff metric. We now show that for large $m$, the diameter of $Y_v$ is less than the Lesbegue number $\delta$.

Since $M_H \geq W_H \geq 2$, we may choose $m$ large enough so that

\[
0 < \frac{N(H)\left(\frac{M_H-1}{W_H-1}\right)}{2^{m-2}} < \delta.
\]

By Lemma 3.2, there exists a number $\kappa(\mathcal{R}^m(G))$ such that if $n > \kappa(\mathcal{R}^m(G))$, then any path which realizes the diameter of $\mathcal{R}^{n+m-1}(G)$ projects to a preminimizing path $\eta$ in $\mathcal{R}^m(G)$. Moreover, for $v \in V$, any path $\xi'$ realizing the diameter of $\mathcal{R}^n(v) \subset \mathcal{R}^{n+m-1}(G)$ projects to a preminimizing path $\xi$ in $\mathcal{R}^m(G)$. Note that $\xi'$ is not necessarily a subset of $\mathcal{R}^n(v)$ and $\xi$ is not necessarily a subset of $\mathcal{R}(v)$.

If $x$ is a replaceable vertex of $\xi$ which is not an endpoint of $\xi$, then the portion of $\xi'$ that passes through $\mathcal{R}^{n-1}(x)$ has endpoints in $\partial \mathcal{R}^{n-1}(x)$ and has length $a(n-1)$; if $x$ is a replaceable vertex of $\xi$ which is an endpoint of $\xi$, then portion of $\xi'$ that passes through $\mathcal{R}^{n-1}(x)$ has an endpoint in $\partial \mathcal{R}^{n-1}(x)$ and has length $b(n-1)$. Hence

\[
N(\xi)a(n-1) \leq \text{diam}(\mathcal{R}^n(v)) \leq L(\xi) + N(\xi)b(n-1),
\]

and

\[
N(\eta)a(n-1) \leq \text{diam}(\mathcal{R}^{m+n-1}(G)) \leq L(\eta) + N(\eta)b(n-1).
\]

By Lemma 3.3 and Inequalities (2) and (3),
Regardless of whether $x$ is a subset of $\mathcal{R}(v)$, we may conclude

\[ N(x) \leq N(H) \quad \text{and} \quad 2^{m-1} \leq N(\eta). \]

Thus, from Inequalities (1), (4), and (5) we have

\[ \lim_{n \to \infty} \frac{\text{diam}(\mathcal{R}^n(v))}{\text{diam}(\mathcal{R}^{n+m-1}(G))} \leq \frac{N(\xi)\left(\frac{M_{H-1}}{W_{H-1}}\right)}{N(\eta)}, \]

Therefore, $\text{diam}(Y_v) < \delta/2$ for all $v \in V$ by Inequality (6).

Let $P$ denote the union of the boundaries of the $Y_v$’s. That is,

\[ P = \bigcup_{v \in V} \partial Y_v. \]

Let $Y_v^o$ denote the interior of $Y_v$. Note that for distinct vertices $v$ and $w$ in $V$, the sets $Y_v^o$ and $Y_w^o$ are pairwise disjoint in $X$. These open sets form the first part of our refinement $\mathscr{B}$ of the open cover $\mathscr{A}$ of $X$. We now construct the remaining elements in our refinement of $\mathscr{A}$, taking care to ensure that they are pairwise disjoint.

Let $d = \inf\{\text{dist}_X(p, q) : p, q \in P, p \neq q\}$. Since for all $v \in V$, the cardinality of $\partial \mathcal{R}^n(v)$ is $|\partial H|$, then $|\partial Y_v| \leq |\partial H|$. Thus $P$ is a finite set and $d > 0$. Let $B_p$ be the open ball in $X$ centered at $p \in P$ with radius $r = \frac{1}{4} \min(\delta, d)$. Then for distinct points $p$ and $q$ in $P$, we have that $B_p \cap B_q = \emptyset$, and the diameter of each ball is less than the Lesbegue number $\delta$ of the open cover $\mathscr{A}$ of $X$.

A nonreplaceable path is defined as a path with no interior replaceable vertices. Let $x \in \mathcal{R}^{m-1}(G) \setminus V$. Then $x$ is nonreplaceable, and any distance realizing path between $x$ and $V$ is nonreplaceable. Thus $\lim_{n \to \infty} \text{dist}_{(\mathcal{R}^{n+m-1}(G), 1)}(\mathcal{R}^n(V), x) = 0$, and $\{Y_v\}_{v \in V}$ forms a cover of $X$. Since $\{Y_v\}_{v \in V}$ forms a cover of $X$ and $Y_v^o = Y_v \setminus P$, the collection $\mathscr{B}$ of open sets in $\{Y_v^o\}_{v \in V}$ together with the open balls in the set $\{B_p\}_{p \in P}$ forms an
open cover of \( X \). Also, no point of \( X \) is contained in any three elements of \( \mathcal{B} \) since the \( Y_v \)'s are pairwise disjoint and the open balls are pairwise disjoint. Finally, the diameter of every element in \( \mathcal{B} \) is less than \( \delta \). Thus \( \mathcal{B} \) is an order 2 refinement of \( \mathcal{A} \) and the topological dimension of \( X \) is one. \( \square \)

5 Examples
We now provide examples of graph substitutions whose limit spaces are fractals. If the graph \( G \) and the replacement rule \( \mathcal{R} \) given by \( H \) are as depicted in Figure 6,
then \((\mathcal{R}^n(G), 1)\) converges in the Gromov-Hausdorff metric to a fractal ruler (shown in Figure 8) which has topological dimension one and Hausdorff dimension \(\frac{\ln 5}{\ln 4}\).

If the graph \(G\) and the replacement rule \(\mathcal{R}\) given by \(H\) are as depicted in Figure 9, then \((\mathcal{R}^n(G), 1)\) converges in the Gromov-Hausdorff metric to the fractal snowflake (shown in Figure 11) which has topological dimension 1 and Hausdorff dimension \(\frac{\ln 5}{\ln 3}\).

Our next example shows a replacement rule whose limit space is a fractal with Hausdorff dimension greater than 2. Figure 12 depicts a graph \(G\) and a replacement rule \(\mathcal{R}\).
given by $H$ such that $(\mathcal{R}^n(G), 1)$ converges to a fractal antenna with topological dimension 1 and Hausdorff dimension $\frac{\ln 5}{\ln 2}$.

We conclude with an example of a fractal which has integer Hausdorff dimension. The Sierpinski tetrahedron (Figure 14), which has topological dimension 1 and Hausdorff dimension 2, is the limit space of $(\mathcal{R}^n(G), 1)$ where $G$ and $H$ are as given in Example 2.
Fig. 14. The Sierpinski tetrahedron

References


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