2.19 Show that any set $A$ is both open and closed relative to itself.

*Note: This problem would be clearer if it said, “... any set $A \subseteq \mathbb{R}^n$ is both ...”*

*Proof.* We first show that $A \subseteq \mathbb{R}^n$ is open relative to itself. Let $x \in A$. We want to show that $x$ is an interior point of $A$. Let $N = D^n(x, r) \cap A$ for some $r > 0$. Then by definition, $N$ is an open neighborhood of $x$ relative to $A$ and $N \subseteq A$. Thus, $x$ is an interior point of $A$ relative to $A$. So $A$ is open relative to itself.

We now show that $A$ is closed relative to itself. We must show that every point in the relative complement of $A$ is an exterior point of $A$ relative to $A$. However, the relative complement of $A$ is $A - A = \emptyset$. So every point in $\emptyset$ satisfies any condition vacuously. Thus, $A$ is closed relative to itself. \qed

2.22 Show that $A$ is open relative to $X$ if and only if $X - A$ is closed relative to $X$.

*Proof.* $X - A$ is closed relative to $X$ if and only if every $x \in X - (X - A)$ is an exterior point of $X - A$ relative to $X$. By Lemma 1 from Relative Neighborhoods Lemmas Worksheet, we have that $X - (X - A) = X \cap A$. However, $A \subseteq X$. So $X - (X - A) = A$.

Thus, $X - A$ is closed relative to $X$ if and only if every $x \in A$ is an exterior point of $X - A$ relative to $X$. That is, if and only if for every $x \in A$, there is a neighborhood $N$ of $x$ relative to $X$ such that $N \subseteq A$. But this is the definition of an interior point of $A$ relative to $X$. Hence, $X - A$ is closed relative to $X$ if and only if every $x \in A$ is an interior point of $A$ relative to $X$. In other words, $X - A$ is closed relative to $X$ if and only if every $A$ is open relative to $X$. \qed
4. Prove that if \( A_i \) is closed for all \( i \), then it is not necessarily true that \( \bigcup_{i=1}^{\infty} A_i \) is closed.

Let \( A_i = \{ x \in \mathbb{R}^n : \|x\| \leq 2 - 1/i \} \). In other words, \( A_i = \text{Cl}(D^n(0, 2 - 1/i)) \). Observe that \( A_i \subseteq A_{i+1} \subseteq D^n(0, 2) \) for all \( i \). So \( \{A_i\} \) is a sequence of larger and larger sets, but \( \lim_{i \to \infty} A_i \) (we haven’t defined this, but you probably understand what I mean.) is not unbounded. Also, \( A_i \) is closed for all \( i \). However, \( \bigcup_{i=1}^{\infty} A_i = D^n(0, 2) \), which is open.