Axioms for a Finite Projective Plane of Order $n$:

Axiom 1: There exists a set of 4 distinct points, no 3 of which are collinear.
Axiom 2: There exists a line with exactly $n+1$ points on it.
Axiom 3: Every pair of distinct points has exactly one line passing through them.
Axiom 4: Every pair of distinct lines has at least one point on both of them.

1.3.7. Let $\ell$ be a line with exactly $n+1$ points $P_1, \ldots, P_{n+1}$ points on it (as guaranteed by Axiom 2). Let $P = P_1$ and let $Q$ be a point not on $\ell$ (as guaranteed by Axiom 1). Show that there is a line $m$ that contains neither $P$ nor $Q$.

Proof. Another way of saying that there is a line $m$ that contains neither $P$ nor $Q$ is to say that there is a line $m$ that doesn’t contain $P$ and $Q$. We proceed by contradiction, carefully negating the statement we want to prove.

Suppose BWOC that every line $m$ contains $P$ or $Q$. By Axiom 1, there is a set of 4 points, no 3 of which are collinear. Thus, there must be a point distinct from $Q$ that is not on $\ell$. Otherwise, every set of 4 points would contain at least 3 (obviously collinear) points from $\ell$ and therefore not satisfy Axiom 1.

Now consider the lines $RP_n$ and $RP_{n+1}$. By our assumption, these lines must pass through $P$ or $Q$. Suppose $RP_n$ passes through $P$. See Figure 1. Since $n > 1$ and $\ell$ has exactly $n+1$ points on it, then $\ell$ has a minimum of 3 points (when $n = 2$). So $P_n$ and $P_{n+1}$ are distinct from $P = P_1$. So $RP_n$ contains the two distinct points $P = P_1$ and $P_n$. However, $\ell$ also contains $P = P_1$ and $P_n$. We know that $\ell \neq RP_n$ because $\ell$ doesn’t contain $R$. Thus, we have two distinct lines ($\ell$ and $RP_n$) that contain two distinct points ($P = P_1$ and $P_n$), which contradicts Axiom 3. So $RP_n$ cannot contain $P$. A similar argument shows that $RP_{n+1}$ cannot contain $P$ either.

Thus, $RP_n$ and $RP_{n+1}$ both must pass through $Q$. See Figure 2. Since $P_n \neq P_{n+1}$, then $RP_n \neq RP_{n+1}$. Otherwise, the distinct lines $RP_n = RP_{n+1}$ and $\ell$ would both contain $P_n$ and $P_{n+1}$, contradicting Axiom 3. Observe that $RP_n$ and $RP_{n+1}$ both contain $R$ and $Q$. But this contradicts Axiom 3.
So it must not be true that every line contains $P$ or $Q$. Hence, there is a line $m$ that doesn’t contain $P$ and doesn’t contain $Q$. That is, there is a line $m$ that contains neither $P$ nor $Q$. \qed

**Axioms for a Finite Affine Plane of Order $n$:**

**Axiom 1:** There exists a set of 4 distinct points, no 3 of which are collinear.

**Axiom 2:** There exists a line with exactly $n$ points on it.

**Axiom 3:** Every pair of distinct points has exactly one line passing through them.

**Axiom 4:** (The Parallel Postulate) Given a line $\ell$ and a point $P$ not on $\ell$, there is exactly one line through $P$ that does not intersect $\ell$.

**1.3.10.** Show that a finite affine plane does not satisfy the principle of duality.

Either ONE of the following proofs will work.

**Proof.**

1. The dual of Axiom 2 says that there is a point with exactly $n$ lines through it. However, Thm 1 from Worksheet 5 says that every point lies on exactly $n + 1$ lines. So the dual of Axiom 2 does not hold. Hence, a finite affine plane does not satisfy the principle of duality.

OR

2. The dual of Axiom 3 says that every pair of distinct lines has exactly one point of intersection. However, we may construct parallel lines (i.e., lines that do not intersect) as follows. By Axiom 2, there is a line $\ell$ with exactly $n$ points on it. By Axiom 1, there is a point $P$, not on $\ell$. By Axiom 4, there is a line $\ell'$ through $P$ and not intersecting $\ell$. So the dual of Axiom 3 does not hold. Hence, a finite affine plane does not satisfy the principle of duality. \qed
1.3.13. Prove that in an affine plane of order $n$, each line contains exactly $n$ points.

**Proof.** Let $\ell$ be an arbitrary line in an affine plane of order $n$. This proof has 3 parts. We first construct a set of points on $\ell$. We then show that $\ell$ has at least $n$ points on it by proving the points in the set are all distinct. Finally, we show that $\ell$ has no more than $n$ points on it.

We now construct a set of points on $\ell$. By Axiom 2, there is a line $m$ with exactly $n$ points. If $m = \ell$, then we are done. So suppose $m \neq \ell$. Thus, there is a point $Q$ on $m$ that is not on $\ell$. By Theorem 1 from Worksheet 5, $Q$ has exactly $n + 1$ lines $\ell_1, \ldots, \ell_{n+1}$ through it. By Axiom 4, exactly one of these lines does not intersect $\ell$. WOLOG, assume $\ell_{n+1}$ does not intersect $\ell$. Thus, $\ell_1, \ldots, \ell_n$ intersect $\ell$ at the points $P_1, \ldots, P_n$, respectively.

**Claim 1:** The points $P_1, \ldots, P_n$ are all distinct.

**Proof.** Suppose BWOC that there are indices $i$ and $j$ with $i \neq j$ such that $P_i = P_j$. WOLOG $P_1 = P_2$. Then $\ell_1$ and $\ell_2$ (which are distinct lines) both pass through $Q$ and $P_1 = P_2$, contradicting Axiom 3. Thus, $P_1 \neq P_2$. So the points $P_1, \ldots, P_n$ are all distinct.

By Claim 1, there must be at least $n$ points on $\ell$.

**Claim 2:** There are no more than $n$ points on $\ell$.

**Proof.** Suppose BWOC that there is another point $P_{n+1}$ on $\ell$. By Axiom 3, there is a unique line $q$ containing both $Q$ and $P_{n+1}$. But as we have seen, $Q$ has exactly $n + 1$ lines $\ell_1, \ldots, \ell_{n+1}$ through it, where $\ell_{n+1}$ is the unique line that does not intersect $\ell$. So $q = \ell_i$, for some $i = 1, \ldots, n$. Then $q = \ell_i$ and $\ell$ both contain the distinct points $P_i$ and $P_{n+1}$, and $\ell \neq q = \ell_i$ because $\ell$ does not contain $Q$. So Axiom 3 is contradicted. Thus, there are no more than $n$ points on $\ell$.

So $\ell$ must have exactly $n$ points on it. Since $\ell$ was an arbitrary line in an affine plane of order $n$, then every line in an affine plane of order $n$ must have exactly $n$ points on it.