**Directional Derivatives**

**Ex 1:** The figure shows the graph of a function \( z = f(x, y) \). The point \( P = (x_0, y_0, f(x_0, y_0)) \) is on this graph. The curve \( CPD \) represents the intersection of the graph of \( f \) with the plane through \( P \) that is parallel to the \( x \)-axis, and the curve \( APB \) represents the intersection of the graph of \( f \) with the plane through \( P \) that is parallel to the \( y \)-axis.

(a) How can we find the slope of the curve \( CPD \) (i.e., the “slope” of the graph of \( f \) at \((x_0, y_0, f(x_0, y_0)) \) in the \( x \)-direction)?

(b) How can we find the slope of the curve \( APB \) (i.e., the “slope” of the graph of \( f \) at \((x_0, y_0, f(x_0, y_0)) \) in the \( y \)-direction)?

**Ex 2:** The figure shows the graph of a function \( z = f(x, y) \). The point \( P = (x_0, y_0, f(x_0, y_0)) \) is on this graph. How can we find the “slope” of the graph of \( f \) at \((x_0, y_0, f(x_0, y_0)) \) in the direction of the unit vector \( \mathbf{u} = (a, b) \)?
**Def:** Let \( z = f(x, y) \) and let \( \mathbf{u} = \langle a, b \rangle \) be a unit vector. The **directional derivative** of \( f \) at \((x_0, y_0)\) in the direction of \( \mathbf{u} \) is given by . . .

**Thm 1:** If the function \( z = f(x, y) \) is differentiable at \((x_0, y_0)\) and \( \mathbf{u} = \langle a, b \rangle \) is a unit vector, then the directional derivative of \( f \) at \((x_0, y_0)\) in the direction of \( \mathbf{u} \) is given by . . .

**Ex 3:** Consider \( f(x, y) = xy^2 \).

(a) Find the directional derivative of \( f \) at \((-3, 1)\) in the direction of \( \mathbf{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle \).

(b) Find the directional derivative of \( f \) at \((-3, 1)\) in the direction of \( \mathbf{v} = \langle 1, -2 \rangle \).
Ex 4: Consider the function \( f(x, y) = \frac{x}{y} \). Let \( P = (1, 2) \) and \( Q = (7, 5) \). Find the directional derivative of \( f \) at \( P \) in the direction of \( Q \).

Applications of the Gradient:

Thm 2: Let \( f \) be differentiable at \((x_0, y_0)\).

1. If \( \nabla f(x_0, y_0) = (0, 0) \), then

\[
D_u f(x_0, y_0) =
\]

2. The direction of maximum increase of \( f \) from \((x_0, y_0)\) is given by . . .

Furthermore, the maximum value of \( D_u f(x_0, y_0) \) over all unit vectors \( u \) is . . .

3. The direction of maximum decrease of \( f \) from \((x_0, y_0)\) is given by . . .

Furthermore, the minimum value of \( D_u f(x_0, y_0) \) over all unit vectors \( u \) is . . .

4. If \( \nabla f(x_0, y_0) \neq (0, 0) \), then . . .
Ex 4: Let $f(x, y) = 6 - 3x^2 - y^2$.

(a) From the point $(1, 2)$, in what direction is $f$ decreasing most rapidly?

(b) How fast is $f$ decreasing in that direction?

(c) Find the level curve of $f$ containing $(1, 2)$, then sketch it along with $\nabla f(1, 2)$. 
Ex 5: Consider the function $f(x, y) = xy$.

(a) Sketch the $z = -3$ level curve.

(b) Verify that $(-1, 3)$ is on this level curve.

(c) Find a unit normal vector to the curve at $(-1, 3)$ and sketch it on the $z = -3$ level curve at $(-1, 3)$. 