EXCEPTIONAL DISCRETE MAPPING CLASS GROUP ORBITS IN MODULI SPACES

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Abstract. Let $M$ be a four-holed sphere and $\Gamma$ the mapping class group of $M$ fixing $\partial M$. The group $\Gamma$ acts on the space $M_B(SU(2))$ of $SU(2)$-gauge equivalence classes of flat $SU(2)$-connections on $M$ with fixed holonomy on $\partial M$. We give examples of flat $SU(2)$-connections whose holonomy groups are dense in $SU(2)$, but whose $\Gamma$-orbits are discrete in $M_B(SU(2))$. This phenomenon does not occur for surfaces with genus greater than zero.

1. Introduction

Let $M$ be a Riemann surface of genus $g$ with $n$ boundary components (circles). Let

$$\{\gamma_1, \gamma_2, ..., \gamma_n\} \subset \pi_1(M)$$

be the elements in the fundamental group corresponding to these $n$ boundary components. Assign each $\gamma_i$ a conjugacy class $B_i \subset SU(2)$ and let

$$B = \{B_1, B_2, ..., B_n\},$$

$$H_B = \{\rho \in \text{Hom}(\pi_1(M), SU(2)) : \rho(\gamma_i) \in B_i, 1 \leq i \leq n\}.$$ 

A conjugacy class in $SU(2)$ is determined by its trace which is in $[-2, 2]$. Hence we might consider $B$ as an element in $[-2, 2]^n$. The group $SU(2)$ acts on $H_B$ by conjugation.

**Definition 1.1.** The moduli space with fixed holonomy $B$ is

$$M_B = H_B / SU(2).$$

Denote by $[\rho]$ the image of $\rho \in H_B$ in $M_B$. The set of smooth points of $M_B$ possesses a natural symplectic structure which gives rise to a finite measure $\mu$ on $M_B$ (see [2, 3]).
Let $\text{Diff}(M, \partial M)$ be the group of diffeomorphisms of $M$ fixing $\partial M$. The mapping class group $\Gamma$ is $\pi_0(\text{Diff}(M, \partial M))$. The group $\Gamma$ acts on $\pi_1(M)$ fixing the $B_i$’s. This action induces a $\Gamma$-action on $\mathcal{M}_B$.

**Theorem 1.2** (Goldman). The mapping class group $\Gamma$ acts ergodically on $\mathcal{M}_B$ with respect to the measure $\mu$.

Since $\mathcal{M}_B$ has a natural topology, one may also study the topological dynamics of the mapping class group action and we have [4, 5]:

**Theorem 1.3.** Suppose $M$ is an orientable surface with boundary and $g > 0$. Let $\rho \in \mathcal{H}_B$ such that $\rho(\pi_1(M))$ is dense in SU(2). Then the $\Gamma$-orbit of the conjugacy class $[\rho] \in \mathcal{M}_B$ is dense in $\mathcal{M}_B$.

In this paper we show:

**Theorem 1.4.** Let $M$ be a four-holed sphere. Then there exists a subset $F \subset [-2, 2]^4$ of two real dimensions with the following property: Suppose $B \in F$. Then there exists $\rho \in \mathcal{H}_B$ with $\rho(\pi_1(M))$ dense in SU(2), but the $\Gamma$-orbit of the conjugacy class $[\rho]$ is discrete in $\mathcal{M}_B$.

Let $G$ be a subgroup of SU(2). We say that a representation $\rho$ is a $G$-representation if $\rho(\pi_1(M)) \subset G$ up to conjugation by SU(2). The group SU(2) is a double cover of SO(3):

$$p : \text{SU}(2) \longrightarrow \text{SO}(3).$$

The group SO(3) contains O(2), and the symmetry groups of the regular polyhedra: $T'$ (the tetrahedron), $C'$ (the cube), and $D'$ (the dodecahedron). Let $\text{Pin}(2), T, C,$ and $D$ denote the groups $p^{-1}(\text{O}(2)), p^{-1}(T'), p^{-1}(C'),$ and $p^{-1}(D')$, respectively. The proper closed subgroups of SU(2) consist of $T, C, D,$ and the closed subgroups of $\text{Pin}(2)$. The group $\text{Pin}(2)$ has two components, and we write

$$\text{Pin}(2) = \text{Spin}(2) \cup \text{Spin}_-(2),$$

where Spin(2) is the identity component of Pin(2).

**Remark 1.5.** Suppose $\rho \in \text{Hom}(\pi_1(M), \text{SU}(2))$. If $\rho(\pi_1(M))$ is not contained in any of the aforementioned closed subgroups, then it is dense in SU(2).

We adopt the following notational conventions: For a fixed representation $\rho$, $X \in \pi_1(M)$, we write $X$ for $\rho(X)$ when there is no ambiguity. A small letter denotes the trace of the matrix represented by the corresponding capital letter.
2. The moduli space of the four-holed sphere

We first review some results that appear in [1, 2, 5]. Suppose $M$ is a three-holed sphere. Then $\pi_1(M)$ has a presentation:

$$\langle A, B, C : ABC = I \rangle,$$

where $A$, $B$, and $C$ represent the homotopy classes of the three boundaries of $M$.

**Proposition 2.1.**

1. A representation $\rho$ on a three-holed sphere is a Spin(2)-representation if and only if $a^2 + b^2 + c^2 - abc - 4 = 0$.
2. A representation $\rho$ on a three-holed sphere is a Pin(2)-representation and not a Spin(2)-representation if and only if $a^2 + b^2 + c^2 - abc - 4 \neq 0$ and at least two of the three: $A$, $B$, $AB$, have zero trace.

**Proof.** See [2, 5].

Suppose $M$ is a four-holed sphere. Then the fundamental group $\pi_1(M)$ admits a presentation

$$\langle A, B, C, D : ABCD = I \rangle.$$

Set $X = AB, Y = BC, Z = CA$. Let $\kappa = (a, b, c, d) \in [-2, 2]^4$ be the holonomies on the boundary. Then the moduli space $M_\kappa$ is the subspace of $[-2, 2]^3$ given by the equation [2, 5]

$$x^2 + y^2 + z^2 + xyz = (ab + cd)x + (ad + bc)y + (ac + bd)z - (a^2 + b^2 + c^2 + d^2 + abcd - 4).$$

**Remark 2.2.** [2] If two representations in $M_\kappa$ share $(x, y, z)$, then they are conjugate.

Let

$$I_{a,b} = \left[ \frac{ab - \sqrt{(a^2 - 4)(b^2 - 4)}}{2}, \frac{ab + \sqrt{(a^2 - 4)(b^2 - 4)}}{2} \right].$$

If $I_{a,b} \cap I_{c,d} \neq \emptyset$, then $M_\kappa$ is a (possibly degenerate) topological sphere (see Figure 1).

The mapping class group $\Gamma$ of the 4-holed sphere is generated by three Dehn twists $\tau_X, \tau_Y, \tau_Z$ [2, 5]. In local coordinates, the actions are

$$\begin{bmatrix} y \\ z \end{bmatrix} \xrightarrow{\tau_X} \begin{bmatrix} ad + bc - x(ac + bd - xy - z) - y \\ ac + bd - xy - z \end{bmatrix},$$

$$\begin{bmatrix} z \\ x \end{bmatrix} \xrightarrow{\tau_Y} \begin{bmatrix} bd + ca - y(ba + cd - yz - x) - z \\ ba + cd - yz - x \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\tau_Z} \begin{bmatrix} cd + ab - z(cb + ad - zx - y) - x \\ cb + ad - zx - y \end{bmatrix}.$$
Consider

$$e^{i\theta} = \left( \begin{array}{cc} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{array} \right), \quad \iota = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

in Pin(2).

**Proposition 3.1.** Suppose $\rho \in \mathcal{H}_{(a,b,c,d)}$ with $a, b, c, d \notin \{\pm 2\}$ and $[\rho] = (x, y, z) \in \mathcal{M}_\kappa$. Then the representation $\rho$ is a Spin(2)-representation
if and only if $x$ is an endpoint of both $I_{a,b}$ and $I_{c,d}$, $y$ is an endpoint of both $I_{b,c}$ and $I_{a,d}$, and $z$ is an endpoint of both $I_{a,c}$ and $I_{b,d}$.

Proof. First, suppose that $\rho$ is a Spin(2)-representation. Then, up to conjugation,

$$
\rho(A) = e^{i\theta_a}, \rho(B) = e^{i\theta_b}, \rho(C) = e^{i\theta_c}, \rho(D) = e^{i\theta_d},
$$

where $\theta_a + \theta_b + \theta_c + \theta_d = 0$. The endpoints of $I_{a,b}$ are given by

$$
\frac{1}{2}(ab \pm \sqrt{(4-a^2)(4-b^2)})
= \frac{1}{2} \cos(\theta_a + \theta_b) \pm \frac{1}{2} \sqrt{(4 - 4 \cos^2(\theta_a))(4 - 4 \cos^2(\theta_b))}
= \frac{1}{2} \cos(\theta_a + \theta_b) \pm |2 \sin(\theta_a) \sin(\theta_b)|
= \cos(\theta_a + \theta_b) \pm \cos(\theta_a - \theta_b) \pm |\cos(\theta_a - \theta_b) - \cos(\theta_a + \theta_b)|.
$$

This implies that an endpoint of $I_{a,b}$ is equal to $2 \cos(\theta_a + \theta_b)$. Similarly, an endpoint of $I_{c,d}$ is equal to $2 \cos(\theta_c + \theta_d)$ which is equal to $2 \cos(\theta_a + \theta_b) = x$. Thus $x$ is equal to an endpoint of both $I_{a,b}$ and $I_{c,d}$. A similar argument shows that $y$ must be an endpoint of $I_{b,c}$ and $I_{a,d}$, and also $z$ must be an endpoint of $I_{a,c}$ and $I_{b,d}$.

To prove the converse, suppose that $\rho$ is such that $x$ is an endpoint of both $I_{a,b}$ and $I_{c,d}$, $y$ is an endpoint of both $I_{b,c}$ and $I_{a,d}$, and $z$ is an endpoint of both $I_{a,c}$ and $I_{b,d}$. Then $2x = ab \pm \sqrt{(4-a^2)(4-b^2)}$ which implies that

$$
4x^2 = a^2b^2 + 16 - 4a^2 - 4b^2 + a^2b^2 \pm 2ab \sqrt{(4-a^2)(4-b^2)}
= a^2b^2 + 16 - 4a^2 - 4b^2 + a^2b^2 \pm 2ab(2x - ab).
$$

Hence

$$
x^2 + a^2 + b^2 - xab = 4
$$

which implies that $\rho$ is a Spin(2)-representation on the three-holed sphere $(A,B,X)$ by Proposition 2.1. Similarly, $(C,D,X)$, $(A,C,Z)$, $(B,D,Z)$, $(A,D,Y)$, and $(B,C,Y)$ are all Spin(2)-representations. As $A,B,C,$ and $D$ all pairwise commute, we have that $\rho$ is a Spin(2)-representation on the entire four-holed sphere. \hfill \square

**Proposition 3.2.** Let $\rho \in \mathcal{H}_\kappa$ and $[\rho] = (x,y,z) \in \mathcal{M}_\kappa$. Suppose $\rho$ is a Pin(2)-representation but not a Spin(2)-representation then one of the following two conditions holds:

1. $\kappa = (0,0,0,0)$,
2. $\kappa = (0,0,c,d)$, where $y = 0$ and $z = 0$, along with the five other symmetric cases.

If $\rho$ satisfies one of the two conditions above, then $\rho$ is a Pin(2)-representation.
Proof. Let \( \rho \) be a Pin(2)-representation but not a Spin(2)-representation. Then at least one of \( A, B, C, \) or \( D \) must be in Spin\( (2) \). However, since \( ABCD = I \), at least two of \( A, B, C, \) or \( D \) must be in Spin\( (2) \). Suppose \( A, B \in \text{Spin}(2) \). If \( C \in \text{Spin}(2) \), then \( D \in \text{Spin}(2) \), then we obtain \( \kappa = (0, 0, 0, 0) \). If \( C \in \text{Spin}(2) \), then \( D \in \text{Spin}(2) \), which implies that \( AC, BC \in \text{Spin}(2) \), i.e., \( y = z = 0 \).

Now consider

\[
A = \iota, B = -\iota e^{i\theta}
\]

which are contained in a Pin(2) subgroup.

Case 1: Let \( \rho \in \mathcal{H}_\kappa \) with \( \kappa = (0, 0, 0, 0) \) with \( x, y, z \) satisfying the equation \( x^2 + y^2 + z^2 + xyz = 4 \). We construct a Pin(2)-representation conjugate to \( \rho \) by setting \( x = 2 \cos \theta \) (in \( A \) and \( B \) above) and setting \( C \) equal to one of \( e^{+i\psi} \iota \), where \( z = -2 \cos \psi \). As \( CA = -e^{\pm i\psi} \) and \( Y = BC \) is either \( e^{i(\theta + \psi)} \) or \( e^{i(\theta - \psi)} \) whose traces are the two solutions of \( x^2 + y^2 + z^2 + xyz = 4 \) for fixed \( x \) and \( z \). Therefore, this Pin(2)-representation is conjugate to \( \rho \).

Case 2: Let \( \rho \in \mathcal{H}_\kappa \) with \( \kappa = (0, 0, c, d) \) with \( y = z = 0 \). Thus \( x, c, d \) satisfy:

\[
x^2 = cdx - c^2 - d^2 + 4 \implies \text{rho restricted to } (X, C, D) \text{ is a Spin(2)-representation by Proposition 2.1.}
\]

We construct a Pin(2)-representation conjugate to \( \rho \) by setting \( x = 2 \cos \theta \) (in \( A \) and \( B \) above) and setting \( C \) to be \( e^{i\psi} \) and \( D = e^{-i(\psi+\theta)} \). As the traces of \( Y = BC \) and \( Z = AC \) are zero, this Pin(2)-representation is conjugate to \( \rho \). \( \Box \)

Propositions 3.1 and 3.2 provide a complete characterization of the Pin(2)-representation classes.

4. Examples

A direct computation shows that the traces of elements in the groups \( C, D \) are in the set \( S = \{0, \pm 1, \pm \sqrt{2}, \pm \frac{\sqrt{5} + 1}{2}, \pm \frac{\sqrt{5} - 1}{2}, \pm 2\} \).

Let \( F \) be the set of \( \kappa = (a, a, c, -c) \in [-2, 2]^4 \) satisfying the following conditions:

1. \( a^2 + c^2 < 4 \),
2. \( a \neq 0 \) and \( c \neq 0 \),
3. \( a \not\in S \) or \( c \not\in S \).

Consider the space \( \mathcal{M}_\kappa \) with \( \kappa \in F \). A direct computation shows

\[
O = \{(a^2 - 2, 0, 0), (2 - c^2, 0, 0)\} \subset \mathcal{M}_\kappa
\]

is \( \Gamma \)-invariant. By condition 1,

\[
I_{a,a} \cap I_{c,-c} = [a^2 - 2, 2] \cap [-2, 2 - c^2] = [a^2 - 2, 2 - c^2]
\]
is a closed interval. Again by condition 1, \( a, c \neq \pm 2 \). Hence Proposition 3.1 implies that elements in \( \mathcal{O} \) do not correspond to Spin(2)-representations. By condition 2, \( a, c \neq 0 \), so the elements in \( \mathcal{O} \) do not correspond to Pin(2)-representations by Proposition 3.2. Finally, by condition 3, they do not correspond to \( \mathcal{C}, \mathcal{D} \)-representations. Thus, by Remark 1.5, the elements in the discrete orbit \( \mathcal{O} \) correspond to representations with dense images in SU(2). This proves Theorem 1.4.

Figure 1 shows one such case with \( \kappa = (\sqrt{2}, \sqrt{2}, \frac{1}{2}, -\frac{1}{2}) \). The special orbit \( \mathcal{O} \) consists of the two points that are intersections of the \( x \)-axis with \( \mathcal{M}_\kappa \), i.e. \( \mathcal{O} = \{ (0, 0, 0), (\frac{7}{4}, 0, 0) \} \). Below is a representation in the conjugacy class \( (0, 0, 0) \in \mathcal{O} \subset \mathcal{M}_\kappa \):

\[
A = B = \begin{bmatrix}
\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i & 0 \\
0 & \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i
\end{bmatrix} \quad \text{and} \quad C = -D = \begin{bmatrix}
\frac{1}{4} + \frac{i}{4} & \frac{\sqrt{14}}{4} \\
-\frac{\sqrt{14}}{4} & \frac{1}{4} - \frac{i}{4}
\end{bmatrix}.
\]

References


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