THE GROWTH OF LIMITS OF VERTEX REPLACEMENT RULES

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Abstract. In this paper, we give necessary and sufficient conditions that determine when a vertex replacement rule given by exactly one replacement graph generates an infinite graph with exponential growth and when it generates an infinite graph with polynomial growth. We also compute the formula for the growth degree of infinite graphs with polynomial growth that are generated by vertex replacement rules given by exactly one replacement graph.

1. Introduction.

Motivated by the study of horospheres on two dimensional branching surfaces (see [?]), the notion of a vertex replacement rule was introduced in [?]. Roughly, a vertex replacement is an iterative rule which replaces certain vertices of a graph with copies of other graphs. These rules generate fractals (see [?]) when the graphs are uniformly scaled, and infinite graphs when the graphs remain unscaled.

In 2006, M. Previte and S.-H. Yang published a paper [?] in which the topological dimension of limits of scaled vertex replacements were computed. The referee for that paper was extraordinarily helpful by suggesting the further investigation of the relationship between the Hausdorff dimension of limits of scaled vertex replacements and the growth degree (also known as the dimension) of limits of unscaled replacements. This paper is the start of that investigation.

In [?] and [?], the Hausdorff dimension of limits of scaled replacements was computed. The main objective for this paper is to study the growth of limits of unscaled replacements. One of our three main theorems shows that under the same conditions given in [?], the growth degree of limits of unscaled replacements is the same formula as the Hausdorff dimension of limits of scaled replacements.

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2. Vertex Replacement Rules.

In this section, we define and provide some basic examples of vertex replacement rules. Throughout this paper, we assume that all graphs are connected, locally finite, unit metric graphs (i.e., each graph is a metric space and every edge is isometric to the interval $(0, 1)$). Furthermore, the distance between two points in a graph is measured by the shortest path in the graph between the two points. More precisely, each graph is treated as a length space.

**Definition 2.1.** A graph $H$ with a designated set of vertices $\{v_1, \ldots, v_k\}$ is called **symmetric about** $\{v_1, \ldots, v_k\}$ if every permutation of $\{v_1, \ldots, v_k\}$ can be realized by an isometry of $H$. The vertices in such a designated set are called **boundary vertices** of $H$, and the set $\{v_1, \ldots, v_k\}$ is denoted by $\partial H$.

The graph $H$ shown in Figure 1 has several possible sets of boundary vertices. For example, $\partial H$ could consist of any pair of vertices, say $v_1$ and $v_2$. Or perhaps $\partial H$ consists of only the single vertex $v_1$. However, $\partial H$ cannot be the set $\{v_1, v_2, v_3, v_4\}$, since $H$ is not symmetric about this set.

**Definition 2.2.** A **vertex replacement rule** $\mathcal{R}$ consists of a finite set of finite graphs (called **replacement graphs**) $H_1, \ldots, H_p$, each with a specified set $\partial H_i$ of boundary vertices, satisfying the requirement that $|\partial H_i| \neq |\partial H_j|$ when $i \neq j$, where $|\cdot|$ denotes the number of vertices in a set.

For example, we can define a vertex replacement rule $\mathcal{R}$ by the replacement graphs $H_1$ and $H_2$ depicted in Figure 2. The boundary vertices designated by $\mathcal{R}$ for each replacement graph are shown with unfilled-in circles (as opposed to solid dots). Note that each replacement graph is symmetric about its set of boundary vertices.

**Definition 2.3.** Let $G$ be a graph, and let $\mathcal{R}$ be a vertex replacement rule given by the replacement graphs $H_1, \ldots, H_p$. A vertex $v$ in a graph
A replacement rule $\mathcal{R}$.

$G$ is replaceable with respect to a replacement rule $\mathcal{R}$ if $\deg(v) = |\partial H_i|$ for some graph $H_i$ in $\mathcal{R}$, where $\deg(v)$ is the degree of $v$.

The replacement rule $\mathcal{R}$ acts on $G$ to produce a new graph $\mathcal{R}(G)$ by substituting for each replaceable vertex $v$ in $G$ a copy of the corresponding replacement graph $H_i$ in such a way that the $\deg(v)$ edges previously adjacent to $v$ in $G$ are now adjacent to the $|\partial H_i|$ boundary vertices of $H_i$. Since $|\partial H_i| \neq |\partial H_j|$ when $i \neq j$, each replaceable vertex has a unique corresponding replacement graph. Also, since each replacement graph $H_i$ is symmetric about $\partial H_i$, it is irrelevant how the edges previously adjacent to $v$ are attached to $H_i$. Thus, vertex replacement is a well-defined procedure.

For example, let $G$ be the graph in Figure 3. With respect to the replacement rule $\mathcal{R}$ in Figure 2, vertices $w_1, w_2$, and $w_3$ are replaceable with $H_1$, and vertices $v_1, v_2$, and $v_3$ are replaceable with $H_2$, but vertices $x_1, x_2$, and $x_3$ are not replaceable. Figure 4 shows $\mathcal{R}(G)$.

Notice that $\mathcal{R}(G)$ in Figure 4 contains vertices of degrees two and three. That is, $\mathcal{R}(G)$ has vertices that are replaceable with respect to $\mathcal{R}$. Thus the replacement rule $\mathcal{R}$ may be iterated to create a sequence of graphs $\mathcal{R}^n(G)$. See Figure 5.
Consider the degree three (and hence replaceable) vertices of $\mathcal{R}(G)$ in Figure 4. Note that each one of these vertices was originally a degree two boundary vertex in a copy of either $H_1$ or $H_2$. On the other hand, consider the boundary vertices in $\mathcal{R} = \{H\}$ from Figure 6. Although these vertices are degree two, once replacement occurs (see Figure 7), they become degree three and are not replaceable. So it is important to keep in mind the fact that once a copy of a replacement graph $H_i$ is inserted into a graph $G$, an edge of $G$ is attached to each boundary vertex and its degree increases by one.
Therefore, to measure the effect of iterating a replacement rule $\mathcal{R}$ on an initial graph $G$, we extend the idea of a replaceable vertex to include the vertices of the replacement graphs themselves. However, one should not treat a replacement graph $H_i$ as an initial graph $G$, but always view it as having already replaced some vertex of a graph $G$. Thus, we have the following definition:

**Definition 2.4.** We say a boundary vertex $v$ of a replacement graph in a replacement rule $\mathcal{R}$ is replaceable with respect to $\mathcal{R}$ if $\deg(v) = |\partial H_i| - 1$ for some graph $H_i$ in $\mathcal{R}$.

For example, in Figure 2 the boundary vertices of $H_1$ are replaceable with $H_2$ (each such vertex will have three edges adjacent after being inserted into a graph $G$), while the remaining vertices of $H_1$ are replaceable with $H_1$. Likewise, the boundary vertices of $H_2$ are replaceable with $H_2$, and the remaining vertices of $H_2$ are replaceable with $H_1$.

There is a natural “crushing map” $\pi$ which will be very useful for the remainder of the paper. This is a pointwise map $\pi: \mathcal{R}(G) \to G$ which undoes replacement by crushing the inserted copies of $H$ to the vertices that they replaced. If $K \subset G$ then $\mathcal{R}(K) \subset \mathcal{R}(G)$ is the subset of $\mathcal{R}(G)$ that corresponds to $K$ after replacement, i.e., $\mathcal{R}(K) = \pi^{-1}(K)$. Note that if $K$ is entirely nonreplaceable, then $\mathcal{R}(K)$ is homeomorphic (but possibly not isometric) to $K$. If $v$ is replaceable by $H_i$ in $G$ then we denote $\mathcal{R}(v) \subset \mathcal{R}(G)$ as the copy of $H_i$ that replaced $v$. In general,
$\mathcal{R}(H_i)$ denotes a copy of $\mathcal{R}^2(v) \subset \mathcal{R}^2(G)$, where $v$ is replaceable by $H_i$ in $G$.

Let $v$ be an $H_i$-replaceable vertex in $G$. For the graph $\mathcal{R}^n(v) \subset \mathcal{R}^n(G)$, define the set $\partial \mathcal{R}^n(v)$ to be the $|\partial H_i|$ vertices $w \in \mathcal{R}^n(v)$ that are adjacent to both $\mathcal{R}^n(v)$ and $\mathcal{R}^n(G) \setminus \mathcal{R}^n(v)$. Set $\mathcal{R}^{n-1}(H_i)$, to be a copy of $\mathcal{R}^n(v) \subset \mathcal{R}^n(G)$, where $v$ is replaceable by $H_i$ and set $\partial \mathcal{R}^{n-1}(H_i) = \partial \mathcal{R}^n(v)$.

Define $N_{H_i}$ to be the number of replaceable vertices of $H_i$. For $|\partial H_i| > 1$, define $B_{H_i}$ to be the minimum number of replaceable vertices on a path connecting distinct boundary vertices. If $|\partial H_i| = 1$, then define $B_{H_i} = 1$ when the boundary is replaceable; otherwise, define $B_{H_i} = 0$.

2.1. Pointed Graphs and Results. In this paper, we will only consider vertex replacement rules given by a single replacement graph $H$, but we note that many of the techniques presented here can be generalized.

We first introduce several functions and three lemmas about those functions. Define the following two functions:

$$a(n) = \text{dist}_{\mathcal{R}^n(v)}(u, u'), \text{ where } u, u' \in \partial \mathcal{R}^n(v) \text{ for } u \neq u'$$

and

$$b(n) = \sup_{z \in \mathcal{R}^n(v)} \{\text{dist}_{\mathcal{R}^n(v)}(\partial \mathcal{R}^n(v), z)\}.$$ 

Let $B_{H}$ denote the minimum number of replaceable vertices on a boundary connecting path in $H$.

Lemma 2.5 is a modified version of Lemma 3.6 in [?].

**Lemma 2.5.** Let $\mathcal{R} = \{H\}$ be a replacement rule and suppose $B_H > 1$. For all $n, m \in \mathbb{N}$, there exist positive constants $\kappa_1$ and $\kappa_2$ such that

$$\kappa_1 \leq \frac{a(n)}{b(n)} \leq \kappa_2, \quad (1)$$

$$\frac{\kappa_1}{B_H^m} \leq \frac{b(n)}{b(n + m)} \leq \frac{\kappa_2}{B_H^m}, \quad (2)$$

and

$$\frac{\kappa_1}{B_H^m} \leq \frac{a(n)}{b(n + m)} \leq \frac{\kappa_2}{B_H^m}. \quad (3)$$

The following lemma is an immediate consequence of Proposition 3.3 from [?].
Lemma 2.6. There are fixed positive constants $K$, $C$, and $\tilde{C}$ (depending on $H$) such that for $n > K$,

$$CB^n H \leq a(n) \leq \tilde{C}B^n H.$$ 

3. Sequences of Replacement Graphs and Their Convergence

Let $G$ be a finite graph with at least one replaceable vertex. There are several possible ways to study the sequences $\{R^n(G)\}$ that result from vertex replacement rules. One approach is to scale each graph $R^n(G)$ in the sequence $\{R^n(G)\}$ so that all graphs have the same diameter and then study the limit of the resulting scaled sequence $\{(R^n(G), 1)\}$ and analyzing limits with respect to the Gromov-Hausdorff metric (see [?]).

In this paper, we will consider pointed graphs (a graph with a distinguished base point). In particular, we will analyze sequences of pointed metric spaces $(R^n(G), x_n)$ where $x_n$ is a sequence of points with each $x_n \in R^n(G)$ with $\pi_n(x_n) = x_{n-1}$, where we insist that $x_0$ not be replaceable in $G$. (This makes replacement a well-defined operation on a pointed graph, since there is a unique $x_1$ so that $\pi(x_1) = x_0$.)

For any pointed metric space $X$ and any $M > 0$, let $B_M(X, x)$ be the ball of radius $M$ centered at $x$. Note that the distance between points in $B_M(X, x)$ may not necessarily be realized by paths within $B_M(X, x)$ as the example below illustrates.

Theorem 3.1. Let $\mathcal{R}$ be a vertex replacement given by one replacement graph $H$ whose boundary vertices are not replaceable. Then for any integer $M > 0$ there exists an $N \in \mathbb{N}$ so that for all $n \geq N$ the balls $B_M(R^n(G), x_n)$ are isometric.

Proof. Let $M > 0$ be given with base point $x_0$ not replaceable in $G$. Consider $R^{4M}(G)$. Clearly, any replaceable vertex must lie at least $4M + 1$ away from $x_{4M}$. Let $p, q$ be in $p$ and $q$ in $(B_M(R^{4M}(G)), x_{4M})$.

Suppose by way of contradiction that there is a replaceable vertex $r$ on a distance realizing path connecting $p$ to $q$. With $d(p, r) \leq d(q, r)$ and $d(p, q) = d(p, r) + d(r, q)$. Then $d(x_{4M}, r) \leq d(x_{4M}, p) + d(p, r)$ which implies that

$$4M + 1 \leq d(x_{4M}, r) \leq d(x_{4M}, p) + d(p, r) \leq M + d(p, r)$$

so $d(p, r) \geq 3M + 1$. But since the path is distance minimizing, $d(p, r) < d(p, q) \leq d(p, x_{4M}) + d(q, x_{4M}) \leq 2M$ which is a contradiction.
This implies that all distances between points in \((B_M(\mathcal{R}^4M(G)), x_4M))\) are realized by paths that contain no replaceable vertices in their interior. Hence after replacement, \((B_M(\mathcal{R}^{4M+1}(G)), x_{4M+1}))\) will be isometric (and graph isomorphic) to \((B_M(\mathcal{R}^{4M}(G)), x_{4M}))\). By induction, for all \(n \geq 4M\) the balls \((B_M(\mathcal{R}^n(G)), x_n))\) are isometric. \(\square\)

**Theorem 3.2.** Let \(\mathcal{R}\) be a vertex replacement given by one replacement graph \(H\) with at least two boundary vertices that are replaceable. Then for any integer \(M > 0\) there exists an \(N \in \mathbb{N}\) so that for all \(n \geq N\) the balls \(B_M(\mathcal{R}^n(G), x_n)\) are isometric.

**Proof.** Let \(M > 0\) be given with base point \(x_0\) not replaceable in \(G\). Let \(NR(G, x_0)\) be the subgraph of \(G\) consisting of all edges and vertices of \(G\) that can be connected to \(x_0\) via a nonreplaceable path. Let \(Q\) be the number of edges in \(NR(G, x_0)\) that are adjacent to replaceable vertices. Then the number of edges in \(NR(\mathcal{R}^n(G), x_n)\) that are adjacent to replaceable vertices is also \(Q\).

Note: It may be the case that \(NR(G, x_0)\) and \(NR(\mathcal{R}(G), x_1)\) are not graph isomorphic (this occurs if two or more edges in \(NR(G, x_0)\) are adjacent to the same replaceable vertex in \(G\)). Further note that \(NR(\mathcal{R}(G), x_1)\) is graph isomorphic to \(NR(\mathcal{R}^2(G), x_2)\), but possibly not isometric.

By Lemma 2.5, we can find \(N\) so that \(a(N) > 2M\). For any \(n \geq N\), let \(X_n\) denote the subgraph of \(\mathcal{R}^n(G)\) consisting of \(NR(\mathcal{R}^n(G), x_n)\) together with \(Q\) copies of \(\mathcal{R}^N(v)\) attached to it, where \(v\) is a replaceable vertex that is in \(\mathcal{R}^{n-N}(G)\) that is in \(NR(\mathcal{R}^{n-N}(G), x_{n-N})\). Clearly, each of the \(X_n\) are isomorphic as graphs.

We claim that any distance minimizing curve in \(B_n(\mathcal{R}^n(G), x_n)\) must not exit \(X_n\). Let \(a, b\) be the endpoints of such a curve. Clearly, \(d(a, x_n) \leq M\) and \(d(b, x_n) \leq M\). If \(c\) is some point outside of \(X_n\) then \(d(c, x_n) > 2M\) which contradicts the fact that \(d(a, b) \leq 2M\) from the triangle inequality, so any distance minimizing curve in \(B_M(\mathcal{R}^n(G), x_n)\) must not exit \(X_n\). Lastly, since all of the \(X_n\) are isomorphic for \(n \geq N\), we have \(B_M(\mathcal{R}^n(G), x_n)\) are all isometric for \(n \geq N\). \(\square\)

The space of all pointed metric spaces \(\mathcal{G}\) is itself a metric space with
\[
\text{dist}_{\mathcal{G}}((G_1, p_1), (G_2, p_2)) = 2^{-M},
\]
where \(M\) is the largest integer such that \((B_M(G_1, p_1), p_1)\) is isometric to \((B_M(G_2, p_2), p_2)\) with an isometry that maps \(p_1\) to \(p_2\).

For our purposes, we will consider only sequences \((\mathcal{R}^n(G), x_n)\) where \(x_0\) itself in \(G\) is not replaceable (which implies that \((\mathcal{R}(G), x_2)\) is well defined).
Lemma 1 Let $x$ in $G$ be a nonreplaceable vertex. Let $Q(x)$ be the number of edges adjacent to replaceable vertices that can be connect to $x$ via a nonreplaceable path.

For a vertex replacement rule $R$ given by a single replacement graph $H$, there exists an $N$ so that for all $n \geq N B_M(R^n(G), x_n)$ are isometric.

In this paper, for any $G$ we start with the inverse system that arises from the surjective crushing maps $\pi_n : R^n(G) \to R^{n-1}(G)$. In particular, for $j > k$ we have surjective maps $\pi_{k,j} = \pi_{k+1} \circ \pi_{k+2} \ldots \pi_j : R^j(G) \to R^k(G)$. The inverse limit will be denoted as $R^\infty(G)$. A point $x$ in the inverse limit is effectively a sequence of points $x_n \in R^n(G)$ with $\pi_n(x_n) = x_{n-1}$. For each $n$, there exists natural maps $\pi_n : R^\infty(G) \to R^n(G)$.

For any pointed metric space $(X, x)$ and any $M > 0$ let $(B_M(x), x)$ be the ball of radius $M$ centered at $x$. We use this metric to define convergence of the sequence $(R^n(G), x_n)$. In particular, if the sequence $(R^n(G), x_n)$ is Cauchy with respect to $\mathcal{G}$, i.e. for any $\epsilon > 0$ there exists an $N$ so that for all $n, m \geq N$

$$\text{dist}_{\mathcal{G}}((R^n(G), x_n), (R^m(G), x_m)) < \epsilon,$$

then by the completeness of $\mathcal{G}$ there is a pointed metric space which we label as $(R^\infty(G), x_\infty)$ to which the sequence converges in $\mathcal{G}$.

Definition 3.3. A marked graph $(G, p)$ is a graph with one designated vertex $p \in G$.

The set $\mathcal{G}$ of isometry classes of all marked graphs is a metric space with

$$\text{dist}_{\mathcal{G}}((G_1, p_1), (G_2, p_2)) = 2^{-M},$$

where $M$ is the largest integer such that the balls of radius $M$ centered at $p_i \in G_i$ are isometric.

For a marked graph $(G, p)$ where $p$ is replaceable, we let $\mathcal{R}(G, p)$ be the set of marked graphs $(\mathcal{R}(G), q)$, where $q$ is any vertex inside the copy of $H_i$ that replaced $p$. For example, if we use the replacement rule in Figure ?? and the initial marked graph in Figure 8, Figure 9 shows two possible marked graphs in the set $\mathcal{R}(G, p_0)$.

Observe that if the marked point $p$ is not replaceable, then the set of marked graphs $(\mathcal{R}(G), q) = \{(\mathcal{R}(G), \mathcal{R}(p))\}$.

Definition 3.4. A sequence satisfying $(\mathcal{R}^i(G), p_i) \in \mathcal{R}(G_{i-1}, p_{i-1})$ for all $i \geq 1$ is called an $\mathcal{R}$-orbit of $(G, p)$. 

Figure 8. A marked graph $(G, p_0)$ where the marked point is designated by a gray vertex.
Proposition 3.5. For every finite graph $G$, the sequence $\{(R^n(G), p_n)\}$ of marked graphs has limit points in $G$.

Proof. The proof of this proposition is a simple generalization of the proof of Proposition 6.1 from [?].

Let $R^\infty(G, p)$ denote the set of limit graphs of all possible $R$-orbits of $(G, p)$. In general, this set consists of more than one element, as illustrated in Figures 10 and 11. Both of these examples use the replacement rule from Figure ?? and both use the initial marked graph in Figure 8. However, their limits are different because of the different choices of marked points in the $R$-orbits of $(G, p_0)$. Note that it is possible for a single $R$-orbit to have more than one limit.

One of the main results of this paper is to show that under the same conditions on the replacement rule $R = \{H\}$ as in Theorem ?? (i.e., replaceable boundary vertices), the limits of any sequence of marked graphs $\{ (R^n(G), p_n) \}$ have growth degree $$\frac{\ln B_H}{\ln N_H}.$$

That is, when the boundary vertices of $H$ are replaceable, the formula for the growth degree of any limit of the sequence $\{ (R^n(G), p_n) \}$ of marked graphs coincides with the Hausdorff dimension of the limit.
of the sequence \(\{(R^n(G), 1)\}\) of scaled graphs. Furthermore, in the last result of this paper, we are able to remove the condition that the boundary vertices of \(H\) are replaceable and compute the growth degree of an infinite graph with polynomial growth that is generated by a vertex replacement rule with nonreplaceable boundary.

4. The Growth of Graphs

In this section, we recall the definitions of exponential and polynomial growth of infinite graphs as well as the growth degree of graphs with polynomial growth. We then state and prove our main results, namely when a vertex replacement rule \(R\) given by exactly one replacement graph generates an infinite graph with exponential growth and when it generates an infinite graph with polynomial growth. Moreover, when \(R\) generates a graph with polynomial growth, we give a formula for its growth degree.

Let \(V(X)\) denote the set of vertices in a graph \(X\).

**Definition 4.1.** The growth function of a locally finite graph \(X\) with respect to a vertex \(x \in V(X)\) is given by

\[
f_X(x, n) = |\{y \in V(X) : d(x, y) \leq n, 0 \leq n\}|
\]

where \(d(x, y)\) denotes the distance between \(x\) and \(y\).

**Definition 4.2.** We say that a locally finite graph \(X\) has exponential growth at a vertex \(x \in V(X)\) if there is a constant \(c > 1\) such that \(f_X(x, n) \geq c^n\) for all \(n \in \mathbb{N}\). Otherwise, \(X\) has non-exponential growth at \(x\). In particular, \(X\) has polynomial growth at \(x \in V(X)\) if there are constants \(c\) and \(d\) such that \(f_X(x, n) \leq cn^d\) for all \(n \in \mathbb{N}\).

**Definition 4.3.** The growth degree at a vertex \(x\) of a graph \(X\) with polynomial growth is

\[
\inf_{d \in \mathbb{R}} \{d : \frac{f_X(x, n)}{n^d} \leq c\}
\]

for some constant \(c\).

In the literature (see [? or [?, §5] for a good summary), the growth of infinite graphs is usually studied in the context of what are called “transitive” graphs. Without getting into the definition of a transitive graph, suffice it to say that for a transitive graph \(X\), the growth function \(f_X\) is independent of the vertex \(x \in X\). Hence, one can define the growth of a transitive graph \(X\). For this paper, we use the following lemma to show that the growth is independent of the vertex \(x \in X\).

**Lemma 4.4.** Let \(X\) be locally finite with \(x, y, \in V(X)\) and \(d(x, y) < \infty\).
(1) If $X$ has exponential growth at $x$ then it does at $y$.
(2) If $X$ has polynomial growth at $x$ then it does at $y$, and the growth degrees are the same.

Proof. Suppose that $X$ has exponential growth at $x$. Let $k = \text{dist}(x, y)$. For $1 \leq k \leq n$,
$$f_X(y, n) \geq f_X(x, n-k) \geq c^{n-k},$$
which implies that $X$ has exponential growth at $y$.

Next, suppose $X$ has polynomial growth at $x$ with growth degree $d$. Then,
$$\frac{f_X(y, n)}{n^d} \leq \frac{f_X(x, n+k)}{n^d} \leq c \left( \frac{n+k}{n} \right)^d.$$
The right most term is bounded, since it converges to $c$ as $n \to \infty$, hence $X$ has polynomial growth at $y$ with growth degree at most $d$. If $X$ had a growth degree at $y$ of $d' < d$ then a symmetric argument to that above would show that the growth degree at $x$ would be at most $d'$. Hence, the growth degrees must agree. \hfill \Box

Henceforth, we will consider replacement rules $R = \{H\}$ given by exactly one replacement graph. Partial results have also been obtained for general replacement rules, but the level of complexity increases very quickly.

Definition 4.5. We say that a replacement rule $R = \{H\}$ is exponential if there exists a nonreplaceable boundary vertex $b \in \partial H$ which can be connected via nonreplaceable paths $\xi_1$ and $\xi_2$ in $H$ to two distinct edges $e_1$ and $e_2$ each adjacent to (not necessarily distinct) replaceable vertices in $H$. We say that a replacement rule $R = \{H\}$ is standard exponential if it is exponential and has the additional condition that the replaceable vertices adjacent to the edges $e_1$ and $e_2$ are distinct.

Remark 4.6. If $|\partial H| > 1$ and $R = \{H\}$ is exponential, then the replacement rule $R'$ with replacement graph $R(H)$ and boundary $\partial R(H)$ is standard exponential.

The example in Figure 12 illustrates Remark 4.6. Note that a replacement rule with replaceable boundary vertices is not exponential.

The next two lemmas prove that the standard exponential replacement rules depicted in Figures 13 and ?? generate infinite graphs with exponential growth.

Lemma 4.7. Let $G$ be any finite graph with a degree one vertex and let $R = \{M\}$, where $M$ consists of simple length $\ell$ paths $\gamma_1$ and $\gamma_2$ which have a common nonreplaceable boundary vertex as one endpoint,
Figure 12. A non-standard exponential replacement rule $\mathcal{R} = \{H\}$ with associated standard exponential rule $\mathcal{R}' = \{\mathcal{R}(H)\}$.

Figure 13. A replacement rule $\mathcal{R} = \{M\}$, where $\ell = 5$ and $k = 2$.

2 distinct replaceable vertices as the other endpoints, and $k$ nonboundary, nonreplaceable vertices in common. Without loss of generality, we may assume that $M$ looks like the graph in Figure 13. (Thus, $M$ has a total of $2\ell - k + 1$ vertices.) Then any limit graph $(X, x)$ of $\mathcal{R}$ has exponential growth.

Proof. Let $(X, x)$ be a limit of the marked graphs $(\mathcal{R}^n(G), p_n)$. There are two cases. Either the marked point $p_i \in \mathcal{R}^i(G)$ is not replaceable for some $i$ or else the marked point $p_i \in \mathcal{R}^i(G)$ is replaceable for all $i$. Let us first consider the case where the marked point $p_i \in \mathcal{R}^i(G)$ is not replaceable for some $i$. Then for all $j > i$, $p_j$ is not replaceable. Thus, the marked point $x$ is an infinite distance from a replaceable vertex. However, observe that for all $j \geq i$, the marked point $p_j \in \mathcal{R}^j(G)$ is always at most a finite distance away (namely $\text{diam}(\mathcal{R}^i(G)) + \ell$) from a degree 3 vertex which projects via $\pi^{j-i}$ to a replaceable vertex in $\mathcal{R}^i(G)$. Therefore, in the limit graph $X$, the marked point $x$ is at most $\text{diam}(\mathcal{R}^i(G)) + \ell$ away from a degree three vertex which is at the base of an infinite binary tree with branch length $\ell$. So to compute $f_{(X,x)}(p,n)$,
we may assume without loss of generality by Lemma 4.4 that \( p \) is one such degree three vertex in \( X \). See Figure 14.

\[
\begin{align*}
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\end{align*}
\]

Figure 14. The limit of the \( \mathcal{R} \)-orbit \((\mathcal{R}^n(v), p_n)\), where \( \mathcal{R} = \{M\} \), \( v \) is a replaceable vertex, and \( p_n \) is not replaceable for some \( n \) is an infinite binary tree with branch length \( \ell \).

Thus for \( n \geq 1 \),

\[
f_{(X,x)}(p, n\ell) \geq 2^n \ell.
\]

Hence, \( f_{(X,x)}(p, n) \geq 2^{n/\ell-1} \ell = \frac{\ell}{2}(2^{1/\ell})^n \) for all \( n \in \mathbb{N} \). So \((X, x)\) has exponential growth.

Next, consider the case where the marked point \( p_i \in \mathcal{R}^i(G) \) is replaceable for all \( i \). Then for any limit graph \((X, x)\) of such an \( \mathcal{R} \)-orbit, the marked point \( x \) is a degree one vertex that is \( \ell - k \) from a unique degree 3 vertex. See Figure 15. Therefore, to compute \( f_{(X,x)}(p, n) \), we may assume by Lemma 4.4 that \( p \) is this degree 3 vertex.

To find a lower bound for the growth function, note that in Figure 15, if one takes the midpoint of a length \( \ell n \) path \((n \text{ even})\) with endpoint
$p$, then one will arrive at the root of three different binary trees each having branch length $\ell$ and total length at least $\frac{n\ell}{2}$. Thus,

$$f_{(X,x)}(p, n\ell) \geq 3\ell 2^{\frac{n}{2}}.$$ 

Hence, $f_{(X,x)}(p, n) \geq \sqrt{2}^{n/\ell - 1} = \frac{1}{\sqrt{2}}[\sqrt{2}^{1/\ell}]^n$ for all $n \in \mathbb{N}$. So $(X, x)$ has exponential growth.

**Figure 15.** A limit graph $(X, x)$ of an $\mathcal{R}$-orbit $(\mathcal{R}^n(v), p_n)$, where $\mathcal{R} = \{M\}$, $v$ is a replaceable vertex, and $p_n$ is replaceable for all $n$.

\[\square\]

**Lemma 4.8.** Let $G$ be any finite graph with a degree one vertex and let $\mathcal{R} = \{K\}$, where $K$ consists of simple length $\ell_1$ and length $\ell_2$ paths $\gamma_1$ and $\gamma_2$, respectively, which have a common nonreplaceable boundary vertex as one endpoint, 2 distinct replaceable vertices as the other endpoints, and $k$ nonboundary, nonreplaceable vertices in common. Without loss of generality, we may assume that $K$ looks like the graph in Figure ???. (Thus, $K$ has a total of $\ell_1 + \ell_2 - k + 1$ vertices.) Then any limit graph $(X, x)$ of $\mathcal{R}$ has exponential growth.

**Proof.** Consider the replacement rule $\mathcal{R}_M = \{M\}$ as given in Figure 13, where $\ell = \ell_1 + \ell_2$. For any $\mathcal{R}$-replaceable vertex $v \in G$ we see that $\mathcal{R}^2(v)$ contains a copy of the replacement graph $M = \mathcal{R}_M(v)$ with the degree one vertices of $\mathcal{R}_M(v)$ mapping to two of the four degree one vertices in $\mathcal{R}^2(v)$ (see Figure ??). In particular, this identification respects the replacement rules $\mathcal{R}_M$ and $\mathcal{R}^2$. Namely, there are one-to-one simplicial maps $j$ and $j'$ so that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{j} & \mathcal{R}^2(v) \\
\downarrow{\mathcal{R}} & & \uparrow{\pi^2} \\
\mathcal{R}_M(M) & \xrightarrow{j'} & \mathcal{R}^4(v)
\end{array}
$$

Thus, any limit graph $(X, x)$ will contain an infinite binary tree of one of the two types in Lemma 4.7. That is, there is a one-to-one