Final Steady Flow near a Stagnation Point on a Vertical Surface in a Porous Medium*

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Abstract

This paper investigates the large time (final steady flow) solutions for unsteady mixed convection boundary layer flow near a stagnation point on a vertical surface embedded in a Darcian fluid-saturated porous medium. Through numerical computations Nazar *et. al.* [1] concluded that for values of the mixed convection parameter $\lambda > -1$, the governing boundary value problem (BVP) had a unique solution. If $\lambda_c \approx -1.4175 < \lambda \leq -1$ two solutions were reported, and if $\lambda < \lambda_c$ then no solutions were found. The purpose of this note is to provide further mathematical and numerical analysis of this problem. We prove existence of a solution to the governing BVP for all $\lambda > -1$. We also present numerical evidence that a second solution exists for $\lambda > -1$, thus giving dual solutions for all $\lambda > \lambda_c$. It is also proven that if $\lambda < -2.9136$ no solution to the BVP exists. Finally, a stability analysis is performed to show that solutions on the upper branch are linearly stable while those on the lower branch are linearly unstable.

Keywords: Boundary value problem, stagnation point, boundary layer, stability

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1 Introduction

We have read with interest the paper by Nazar, *et al.* [1] on unsteady mixed convection stagnation point flow on a vertical surface in a fluid-saturated porous medium. Such problems have application to convective transport processes around deep geological repositories for the disposal of high-level nuclear waste. The main concern of their work centered on the time-dependent behavior in the neighborhood of the stagnation point on a vertical wall. However, we were attracted by the steady flow results calculated using the Keller-box method and displayed in their figure 2. This figure exhibits the variation of the shear stress parameter $G''(0)$ as a function of the mixed convection parameter λ and reveals dual solutions in the parameter range $\lambda_c < \lambda < -1$, where $\lambda_c = -1.417$. A primary thrust of the present investigation is to prove existence of these solutions.

A secondary interest in their work concerns the end point G^* to which the lower branch of the parametric curve apparently asymptotes: $G''(0)\ to G^*$ as $\lambda \to -1$, "where the exact value of G^* cannot be determined." Our experience with boundary-layer problems of this type generally reveals a focal point located at the origin in $G''(0) - \lambda$ space; see for instance, Riley and Weidman [2] and Weidman, *et al.* [3] for Newtonian boundary layer examples. Our integrations, using a standard shooting technique, shows that the lower branch does not terminate at $\lambda = -1$, but continues indefinitely to large values of $\lambda > 0$. A final thust of the present work is to investigate the stability of the dual solutions to ascertain whether one or both are to be expected in practice.

The outline is as follows. The steady and unsteady (different than in [1]) equations governing this mixed convection porous medium flow are derived in §2. Existence of a solution for $\lambda > -1$ is proven in §3 and qualitative properties of that solution are given in §4. Evidence for a second nonmonotonic solution for $\lambda > -1$ found numerically and reported in §5 and nonexistence results for $\lambda < -2.9136$ is proven in §6. The paper is concluded in §7 with an analysis of the stability of the dual solutions.

2 Unsteady Porous Media Equations

The equations of fundamental interest are the steady porous media equations for mixed convection boundary layer flow near a stagnation point on a vertical impermeable surface. We will be further concerned with the stability of the dual steady solutions. This requires the analysis of an unsteady equation different from that reported by Nazar, *et al.* [1]. The derivation of both the steady and unsteady equations used in this study are presented here for future reference.

The mixed convection porous medium problem in [1] takes the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{2.1}$$

$$u(x, y) = \frac{U_e}{L} x + \frac{gK\beta}{\nu}(T - T_\infty) \tag{2.2}$$

$$\sigma \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2}. \tag{2.3}$$

where x, y are downstream and plate normal coordinates with respective velocity components u, v, U_e/L is the strain rate of the stagnation flow, T is temperature, t is time and g, K, β ν σ and α_m are constants.

In the present analysis there is no need to follow Williams and Rhyne [3] who found a similarity variable involving both y and t. Instead we only require the simpler formulation

$$u = \frac{U_e}{L} x f'(\eta, \tau), \qquad v = -\left(\frac{U_e \alpha_m}{L}\right)^{1/2} f(\eta, \tau), \qquad T = T_\infty + \frac{sT_0}{L} x\, \theta(\eta, \tau) \tag{2.4}$$

$$\eta = \left(\frac{U_e}{L\alpha_m}\right)^{1/2} y, \qquad \tau = \frac{U_e}{\sigma L} t \tag{2.5}$$

to obtain

$$\theta_\tau + f'\theta - f\theta' = \theta'' \tag{2.6}$$

$$f' = 1 + \lambda\theta \tag{2.7}$$

where $\lambda = s\, gK\beta T_0/\nu U_e$ is the mixed convection parameter relating buoyancy forces to the strength of the stagnation flow. Primes denote differentiation with respect to η and the subscript denotes differentiation with respect to τ.

Elimination of θ gives the partial differential equation governing $f(\tau, \eta)$, *viz.*,

$$f''' + ff'' - f'^2 + f' - f'_\tau = 0. \tag{2.8}$$

Analysis of the existence and nonexistence of solutions of the steady flow problem is taken up in §§3, 4, 5, 6 and a stability analysis utilizing the unsteady equation is presented in §7.

3 Existence of a solution for $\lambda > -1$

Let $f = F(\eta)$ be the solution to the steady flow problem with parameter λ. Then the steady BVP is given by

$$F''' + FF'' + F' - F'^2 = 0 \tag{3.1}$$

subject to

$$F(0) = 0, \quad F'(0) = 1 + \lambda, \quad F'(\infty) = 1. \tag{3.2, 3, 4}$$

Theorem 1. For any $\lambda > -1$ there exists a solution to the BVP (3.1-4).

To study existence of a solution to the BVP (3.1-4) we will consider a related initial value problem (IVP); (3.1-3) along with

$$F''(0) = \alpha \tag{3.5}$$

where α is a free parameter. We will denote the solution of this IVP by $F(\eta; \alpha)$. Occasionally the dependence on η or α or both will be dropped for notational convenience. We will use a topological shooting argument to show that α can be chosen so that the solution of the IVP exists for all $\eta > 0$ and also satisfies (3.4), giving a solution to the BVP. This argument will involve two cases; $\lambda > 0$ and $-1 < \lambda < 0$. (If $\lambda = 0$, then a trivial solution to the BVP is given by $F'(\eta) \equiv 1$. Also, as the two cases are similar only the case $\lambda > 0$ is presented in detail.)

The existence proof for $\lambda > 0$ will involve the following subsets of $(-\infty, 0)$:

$$\mathcal{A} = \{\alpha < 0 \mid F''(\eta; \alpha) = 0 \text{ strictly before } F'(\eta; \alpha) = 1\}$$

and

$$\mathcal{B} = \{\alpha < 0 \mid F'(\eta; \alpha) = 1 \text{ strictly before } F''(\eta; \alpha) = 0\}.$$

The next two lemmas will show that these two sets are non-empty and open.

Lemma 1. The set $\mathcal{A}$ is non-empty and open.

Proof. We will show that for all $\alpha < 0$, $|\alpha|$ sufficiently small, $\alpha \in \mathcal{A}$. Consider $\alpha = 0$. Since $\lambda > 0$, from (3.1) we have that $F'''(0; 0) = \lambda(\lambda + 1) > 0$. Thus, from $F'(0; 0) = 1 + \lambda$, $F''(0; 0) = 0$ and $F'''(0; 0) > 0$ we can conclude that there exists an $\varepsilon > 0$ such that $F'(\eta; 0) > 1$ and $F''(\eta; 0) > 0$ for all $\eta \in (0, \varepsilon]$. By continuity of the solutions of the IVP in its initial conditions on bounded intervals, we can choose $\alpha < 0$, $|\alpha|$ sufficiently small so that $F'(\eta; \alpha) > 1$ for all $\eta \in [0, \varepsilon]$ and $F''(\varepsilon; \alpha) > 0$. But $F''(0; \alpha) = \alpha < 0$. Thus there exists a first $\eta_0 \in (0, \varepsilon)$ such that $F''(\eta_0; \alpha) = 0$ with $F'(\eta; \alpha) > 1$ for all $\eta \in [0, \eta_0]$. Thus $\mathcal{A}$ is non-empty.

To show that $\mathcal{A}$ is open, consider $\overline{\alpha} \in \mathcal{A}$. We will show that all α sufficiently close to $\overline{\alpha}$ are also in $\mathcal{A}$. At η_0, $F''(\eta_0; \overline{\alpha}) = 0$ and for all $\eta \in [0, \eta_0]$ we have $F'(\eta; \overline{\alpha}) > 1$. Evaluating (3.1) at η_0 implies that

$$F'''(\eta_0; \overline{\alpha}) = F'(\eta_0; \overline{\alpha}) \left(F'(\eta_0; \overline{\alpha}) - 1\right) \neq 0.$$

Thus, by continuity of the solutions of the IVP in its initial conditions, for α sufficiently close to $\overline{\alpha}$, $F''(\eta; \alpha)$ will also have a root near η_0 with $F'(\eta; \alpha) > 1$ for all η up to this root. Thus $\alpha \in \mathcal{A}$ and $\mathcal{A}$ is open.

Lemma 2. The set $\mathcal{B}$ is non-empty and open.

Proof. First note that integrating (3.1) from 0 to η gives:

$$F''(\eta) = \alpha - F(\eta) + \int_0^\eta F'(t)^2 dt - \int_0^\eta F(t) F''(t) dt.$$

Integrating the last term by parts results in

$$F''(\eta) = \alpha - F(\eta) \left(F'(\eta) + 1\right) + 2 \int_0^\eta F'(t)^2 dt. \tag{3.6}$$

We will show that for $\alpha < 0$, $|\alpha|$ sufficiently large, then $\alpha \in \mathcal{B}$. We claim that for such α, $F' = 1$ in the interval $[0, 1]$ strictly before $F'' = 0$. Suppose that the assertion is false. Then one of the following must occur: (i) $F'' = 0$ at some first point in [0,1] with $F' > 1$, (ii) $F'' < 0$ and $F' > 1$ for all $\eta \in [0, 1]$, or (iii) $F'' = 0$ and $F' = 1$ simultaneously. We eliminate each of these in turn. To begin with (i), suppose that there exists a first $\eta_1 \in [0, 1]$ with

$$F''(\eta_1) = 0 \tag{3.7}$$

with $1 < F'(\eta) \leq \lambda + 1$ for $\eta \in [0, \eta_1]$. Integrating this inequality from 0 to η gives $\eta \leq F < (\lambda + 1)\eta$. Using these bounds on F and F' in (3.6) we conclude that

$$F''(\eta) \leq \alpha + 2(\lambda + 1)^2 \quad \forall \eta \in [0, \eta_1].$$

Thus if we choose $\alpha < -2(\lambda + 1)^2$ then $F''(\eta_1) < 0$ contradicting (3.7). A similar argument shows that if $\alpha < -2(\lambda + 1)^2 - \lambda$ then we cannot have (ii) $F'' < 0$ and $F' > 1$ on all of $[0, 1]$. (i.e. $F'(1)$ will be less than 1.) This leaves only the case (iii) $F' = 1$ and $F'' = 0$ simultaneously; however, substituting this information into (3.1) gives $F''' = 0$ implying that $F'(\eta) \equiv 1$, contradicting the basic existence and uniqueness theorem for initial value problems, as $F'(0) = \lambda + 1 \neq 1$. Thus if $\alpha < -2(\lambda+1)^2 - \lambda$ then we must have $F' = 1$ strictly before $F'' = 0$ and therefore $\alpha \in \mathcal{B}$. An argument similar to that of Lemma 1 shows that $\mathcal{B}$ is also open.

Thus by lemmas 1 and 2, the sets $\mathcal{A}$ and $\mathcal{B}$ are non-empty and open. They are also obviously disjoint. But the interval $(-\infty, 0)$ is connected and thus $\mathcal{A} \cup \mathcal{B} \neq (-\infty, 0)$. Therefore, there exists some α^* such that $\alpha^* \notin \mathcal{A}$ and $\alpha^* \notin \mathcal{B}$. (i. e. we cannot have $F'' = 0$ strictly before $F' = 1$ and we cannot have $F' = 1$ strictly before $F'' = 0$.) As previously observed, we cannot have $F' = 1$ and $F'' = 0$ simultaneously, thus the only other possibility is $F'(\eta;\alpha^*) > 1$ and $F''(\eta;\alpha^*) < 0$ for all $\eta > 0$. From (3.1) we see that we must then have $F'(\infty;\alpha^*) = 1$ giving the existence of a solution to the BVP for $\lambda > 0$. The argument for $-1 < \lambda < 0$ is similar and the theorem is proved.

4 Qualitative properties of the solution

The analysis of the previous section showed that for $\lambda > 0$ a solution exists with the property $F''(\eta) < 0$ for all $\eta > 0$. Thus $F'(\eta)$ is monotonic and we can further conclude that $1 < F'(\eta) < \lambda + 1$ and $F(\eta) > 0$ for all $\eta > 0$. Using an argument given by McLeod and Rajagopal [4] we can conclude that for $\lambda > 0$ there cannot be two solutions with the property that $F'(\eta)$ is monotonic.

Suppose for contradiction that there were two monotonic solutions, F_1' and F_2'. If we let $\phi = F_1 - F_2$ then ϕ satisfies

$$\phi''' + F_2\phi'' - (F_1' + F_2' - 1)\phi' + F_1''\phi = 0 \tag{4.1}$$

subject to

$$\phi(0) = 0, \quad \phi'(0) = 0, \quad \phi'(\infty) = 0. \tag{4.2}$$

Suppose without loss of generality that $\phi''(0) > 0$. Then initially $\phi > 0$, $\phi' > 0$, $\phi'' > 0$, and so long as these inequalities are maintained,

$$\phi'' \exp\left(\int_0^\eta F_2(t)dt\right)$$

is increasing since $F_1' + F_2' - 1 > 0$ and $F_1'' < 0$. Hence ϕ, ϕ', ϕ'' never vanish, which contradicts $\phi'(\infty) = 0$.

5 A second nonmonotonic solution for $\lambda > -1$

The result of the previous section indicates that for $\lambda > 0$ at least, if a second solution exists, it cannot be monotonic. In order to investigate the possible existence of such solutions a numerical shooting method was applied to the BVP using the fourth order Runge-Kutta scheme.

It was found that two solutions exist for all $\lambda > \lambda_c \approx -1.4175$ with the two solution branches coalescing at λ_c. This is contrast with Nazar *et. al.* [1] who report two solutions only for $\lambda_c < \lambda \leq -1$. Figure 1 plots $F''(0)$ as a function of λ, (cf. figure 2 in [1]). As in [1] we denote the upper branch by $F_1''(0)$ and the lower branch by $F_2''(0)$. For the upper solution branch, $F_1'(\eta)$ is always monotonic; decreasing for $\lambda > 0$ and increasing for $\lambda_c < \lambda < 0$. On the lower solution branch, $F_2'(\eta)$ is monotonic if $\lambda_c < \lambda <\approx -1.3785$ and nonmonotonic if $\lambda >\approx -1.3785$. The functions $F_1'(\eta)$ and $F_2'(\eta)$ are plotted for various values of λ in Figure 2.

As can be seen from (3.1), F' can only have a minimum if $F' < 0$ or $F' > 1$. Conversely, F' can only have a maximum in the range $0 < F' < 1$. Thus a solution to the BVP cannot have an extremum above $F' = 1$ and any nonmonotonic solution must have at least one negative minimum. From our numerical investigation it appears that all nonmonotonic solutions have precisely one extremum, a minimum below $F' = 0$.

6 Nonexistence results

Theorem 2. If $\lambda < -2.9136$, then no solution to the BVP exists.

Proof. For the sake of contradiction suppose a solution exists. Then since $F(0) = 0$ and $F'(0) = \lambda + 1 < 0$, we have $F(\eta) < 0$ initially. But since $F' \to 1$ we must ultimately have $F \to \infty$. Thus there exists a first η_2 such that $F(\eta_2) = 0$. Obviously $F'(\eta_2) \geq 0$, but also notice that we must have $F'(\eta_2) < 1$, since F' cannot have a maximum at or above 1. Multiplying (3.1) by F'' and integrating from 0 to η_2 gives

$$F''(\eta_2)^2 = \alpha^2 - 2\int_0^{\eta_2} F(t)F''(t)^2 dt + \frac{1}{3}F'(\eta_2)^2\left(2F'(\eta_2) - 3\right) + \frac{1}{3}(\lambda+1)^2(1-2\lambda).$$

Using the facts the $F(\eta) \leq 0$ on $[0, \eta_2]$ and $0 \leq F'(\eta_2) < 1$ we obtain the bound

$$F''(\eta_2)^2 > -\frac{1}{3}\lambda^2(2\lambda+3)$$

from which we can conclude that either

$$F''(\eta_2) > \sqrt{-\frac{1}{3}\lambda^2(2\lambda+3)} \tag{6.1}$$

or

$$F''(\eta_2) < -\sqrt{-\frac{1}{3}\lambda^2(2\lambda+3)}. \tag{6.2}$$

We will show that both of these possibilities lead to contradictions. Beginning with (6.2), if $F''(\eta_2) < 0$, then integrating (3.1) from 0 to η_2 leads us to conclude that

$$\alpha = F''(\eta_2) - 2\int_0^{\eta_2} F'(t)^2 dt < 0.$$

Thus since $F''(0) = \alpha < 0$, F' is initially decreasing and must therefore have a first minimum, at some $\eta^* < \eta_2$, with $F(\eta^*) < 0$ and $F'(\eta^*) < -1$. This last is true since $F'(0) = \lambda + 1 < -1$ by our assumption on λ and F' is decreasing until its minimum at η^*. Integrating (3.1) from η^* to η_2 we conclude that

$$F''(\eta_2) = F(\eta^*)\left(F'(\eta^*) + 1\right) + 2\int_{\eta^*}^{\eta_2} F'(t)^2 dt > 0$$

contradicting (6.2).

So next assume that (6.1) holds and thus $F''(\eta_2) > 0$. Then we claim that $F'' > 0$ on the interval $[\eta_2, \eta_2 + 1/\sqrt{2}]$. For if F'' had a root in this interval, at η_3 say, then integrating (3.1) from η_2 to η_3 gives

$$F(\eta_3)\left(F'(\eta_3) + 1\right) = F''(\eta_2) + 2\int_{\eta_2}^{\eta_3} F'(t)^2 dt. \tag{6.3}$$

On $[\eta_2, \eta_3]$, $0 \le F' < 1$ and on integration we conclude that $0 \le F < \eta_3 - \eta_2$. Using this along with (6.1) in (6.3) gives

$$\eta_3 - \eta_2 > \frac{1}{2}\sqrt{-\frac{1}{3}\lambda^2(2\lambda+3)}$$

which is greater than $1/\sqrt{2}$ if $\lambda <\approx -2.14935$. But by assumption $\lambda < -2.9136$ and so we can conclude that $F''(\eta) > 0$ on $[\eta_2, \eta_2 + 1/\sqrt{2}]$.

Finally integrating (1.1) from η_2 to $\eta \in [\eta_2, \eta_2 + 1/\sqrt{2}]$ we get

$$F''(\eta) = F''(\eta_2) - F(\eta)\left(F'(\eta) + 1\right) + 2\int_{\eta_2}^{\eta} F'(t)^2 dt.$$

Using the bounds $0 \le F' < 1$, $F \ge 0$ and (6.1) we obtain

$$F''(\eta) > \sqrt{-\frac{1}{3}\lambda^2(2\lambda+3)} - \sqrt{2}, \quad \forall \eta \in [\eta_2, \eta_2 + 1/\sqrt{2}]. \tag{6.4}$$

If the right hand side of (6.4) is greater than $\sqrt{2}$, then on integration we will have $F'(\eta_2 + 1/\sqrt{2}) > 1$ and therefore F' cannot be a solution to the BVP. The right hand side of (6.4) will be greater than $\sqrt{2}$ when λ is less than the root of $-2\lambda^3 - 3\lambda^2 - 24 = 0$, which occurs at $\lambda \approx -2.9136$. Thus if $\lambda < -2.9136$, (6.1) also leads to a contradiction and thus no solution exists and the theorem is proved.

7 Stability analysis

Our numerical results reveal that the lower branch solution continues well beyond the point $\lambda = -1$ reported in [1]. It is of interest to ascertain the stability of these dual solutions which apparently exist for all $\lambda > -1.4175$. To this end we return to the unsteady form of the problem derived in §2 and test the stability of the steady solutions following Merkin [5]

$$f(\eta, \tau) = F(\eta) + e^{-\gamma\tau} g(\eta) \tag{7.1}$$

where g and all its derivatives are small compared to the steady solution F and its derivatives. Inserting the posited solution form in (2.8) and linearizing yields the equation

$$g''' + Fg'' + (1 + \gamma - 2F')g' + F''g = 0 \tag{7.2}$$

governing eigenfunctions $g(\eta)$ and corresponding eigenvalues γ. Since Eq. (7.2) satisfies homogeneous boundary and far-field conditions

$$g(0) = 0, \qquad g'(0) = 0, \qquad g'(\infty) = 0 \tag{7.3}$$

one may set $g''(0) = 1$ without loss of generality. At each value of λ, stability is determined by the sign of the smallest eigenvalue γ_1, with $\gamma_1 > 0$ representing a stable solution and $\gamma_1 < 0$ representing an unstable solution.

A search for the lowest eigenvalues γ_1 satisfying (7.2) and (7.3) was carried out and the results are plotted in Figure 3. Clearly the upper branch solutions are positive, the lower branch solutions are negative and $\gamma_1 \to 0$ as the turning point $\lambda_c = -1.417$ is approached from the right. We conclude that of the dual steady flow solutions, the upper branch solutions are linearly stable while the those on the lower branch are linearly unstable.

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Figure Captions

FIGURE 1: Reduced skin friction $F''(0)$ as a function of λ.

FIGURE 2: Self-similar velocity profiles $F_1'(\eta)$ and $F_2'(\eta)$ for selected values of λ: (a) $\lambda = -1.4$ for which $F_1''(0) = 0.36073$ and $F_2''(0) = 0.043513$, (b) $\lambda = -.5$ for which $F_1''(0) = 0.51435$ and $F_2''(0) = -0.39553$, and (c) $\lambda = 10$ for which $F_1''(0) = -34.081085$ and $F_2''(0) = -36.455849$.

FIGURE 3: Plot of lowest eigenvalues γ_1 as a function of λ, showing positive values as solid circles and negative values as open circles.

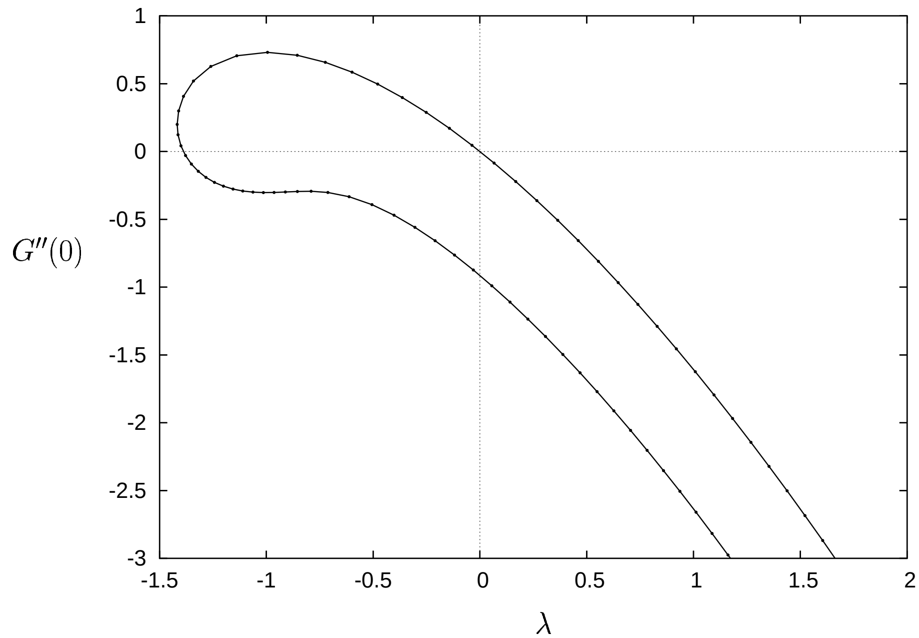

Figure 1

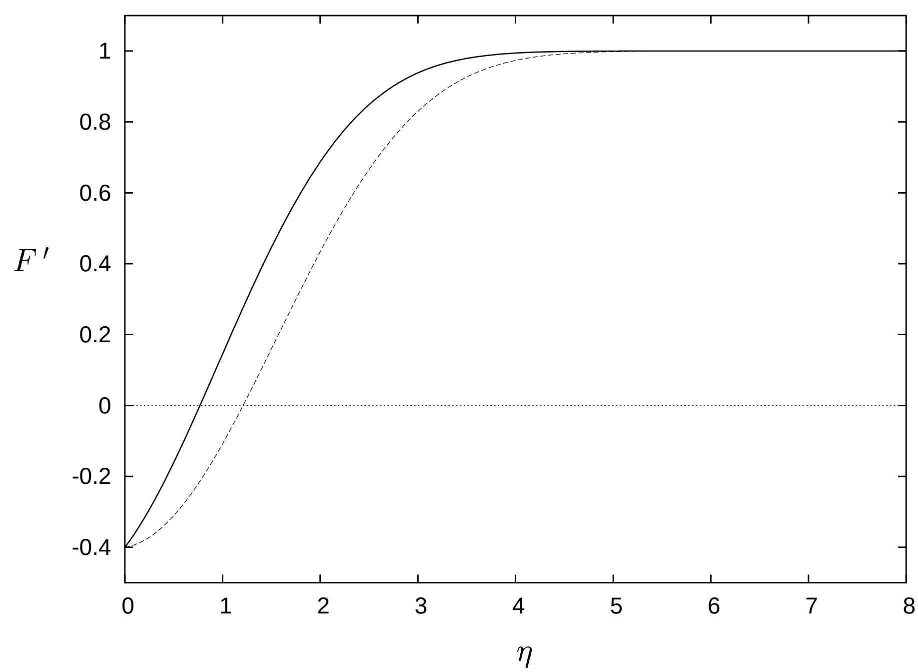

Figure 2a