§4 and §8 Hints

• Here are some common guidelines for proofs that involve sup \( A \)
  (a) Suppose you are trying to prove
  \[
  \sup A \leq x,
  \]
  where \( x \) is some number.
  One strategy is to show that \( x \) is an upper bound of \( A \).
  (Then the result follows since \( \sup A \) is the least upper bound of \( A \).)
  To do this: Let \( a \in A \) be any element of \( A \) and show that \( x \geq a \).
  
  (b) Similarly, when trying to prove \( \inf A \geq x \), where \( x \) is some number.
  One strategy is to show that \( x \) is a lower bound of \( A \).
  (Then the result follows since \( \inf A \) is the greatest lower bound of \( A \).)
  To do this: Let \( a \in A \) be any element of \( A \) and show that \( x \leq a \).
  
  (c) If you ever know that \( \sup A > y \) for some number \( y \),
  then sometimes it is helpful to remember that there must exist an \( a \in A \) so that
  \( \sup A \geq a > y \).

• Here are some tricks that may come up in \( \epsilon/N \) proofs.
  (a) When trying to solve
  \[
  |a_n - L| < \epsilon \text{ for } n > junk(\epsilon)...
  \]
  You can use an intermediate term in between \( |a_n - L| \) and \( \epsilon \) that can be solved directly for \( n \).
  (Note: This term should have some form of \( n \) in the denominator, not the numerator!)
  e.g.
  \[
  \frac{1}{n} e^{-n} \leq \frac{1}{n}
  \]
  since \( e^{-n} < 1 \) for \( n \geq 1 \).
  
  e.g.
  \[
  \frac{1}{n^2 + 2n + 11} < \frac{1}{n^2}
  \]
  (Since \( n^2 < n^2 + 2n + 11 \))
  
  e.g.
  \[
  \frac{1}{n^2 - 2n + 11} = \frac{1}{(n - 1)^2 + 10}
  \]
By insisting that \( n > 11 \) we may write
\[
\frac{1}{n^2 - 11} < \frac{1}{n^2 - n} = \frac{1}{n(n - 1)} \leq \frac{1}{n}
\]

additionally, we could have done:
By insisting that \( \frac{n^2}{2} > 11 \) we may write
\[
\frac{1}{n^2 - 11} < \frac{1}{n^2 - \frac{n^2}{2}} \leq \frac{1}{\frac{n^2}{2}} = \frac{2}{n^2}
\]

(we can insist that \( n \) be big enough so that \( 2n < \frac{n^2}{4} \) and \( 100 < \frac{n^2}{4} \)
then
\[
\frac{1}{n^2 - 2n - 100} < \frac{1}{n^2 - \frac{n^2}{4} - \frac{n^2}{4}} \leq \frac{1}{\frac{n^2}{2}} = \frac{2}{n^2}
\]

HW:

(1) Let \( S \) be a nonempty bounded subset of \( \mathbb{R} \)
Let \( T = \{2s : s \in S\} \) (essentially, \( T \) is obtained by doubling the elements of \( S \))
Prove that \( \sup T = 2 \sup S \).

(2) Use the \( \epsilon/\ N \) definition to prove that
\[
a_n = \frac{n^2}{n^2 - 4n - 11}
\]
converges

(3) Assume that \( a_n \) converges to \( L \).
(a) Prove that \( |a_n| \) converges to \( |L| \).
(b) Is the converse true? Prove or give a counterexample.