10. Solve \( y'' + 2y' + 4y = f(t), y(0) = 0, \ y'(0) = 0 \), where \( f(t) \) is given in the previous problem.

11. Graph the function \( f(t) = t - (2t - 2)u(t - 1) + (2t - 4)u(t - 2) - (2t - 6)u(t - 3) + \ldots \).

12. Solve \( y'' + 2y' + 4y = f(t), y(0) = 0, \ y'(0) = 0 \), where \( f(t) \) is given in the previous problem.

13. Consider \( f(t) = e^{2t} \) made into a periodic function \( \tilde{f}(t) \) by taking \( f_T(t) \) where \( T = 1 \).
   (a) Plot \( \tilde{f}(t) \) for \( 0 < t < 4 \).
   (b) Find \( \mathcal{L}[\tilde{f}(t)] \)
   (c) \( y'' + 2y' + 3y = \tilde{f}(t), y(0) = 0, \ y'(0) = 0 \),

14. Use the differentiation theorem to verify that \( \mathcal{L}[t \ u(t-a)] = e^{-as} \frac{1}{s^2} \)

15. Use appropriate theorems to compute \( \mathcal{L}[t \sin(t) u(t-a)] \)

5.5 Convolution and the Laplace Transform

We introduce a new operation \( * \) between two functions called the convolution.

\[
\text{Convolution of Two Functions}
\]

Let \( f(t) \) and \( g(t) \) be two functions. Define a new function:

\[
f * g(t) = \int_0^t f(t-w)g(w) \, dw
\]

Note that \( f * g \) is itself a function of \( t \). Moreover note that if we substitute \( v = t - w \) then \( dv = -dw \) and the integral becomes

\[
\int_{w=t}^{w=0} f(v)g(t-v)(-1) \, dw = \int_{w=0}^{w=t} f(v)g(t-v) \, dw
\]

which is \( g * f \).

Example 5.29 Find \( t * 1 \)
5.5. CONVOLUTION AND THE LAPLACE TRANSFORM

Solution: Since \( f \ast g = g \ast f \), we can compute \( 1 \ast t \) easier, so let \( f(t) = 1 \) and \( g(t) = t \). Then

\[
f \ast g = \int_0^t f(t-w)g(w) \, dw = \int_0^t w \, dw = \frac{w^2}{2} \bigg|_0^t = \frac{t^2}{2}
\]

\( \Box \)

We see that convolution is not the same as regular multiplication.

Example 5.30 Find \( t \ast e^t \)

Solution: We set \( f(t) = t \) and \( g(t) = e^t \).

Then

\[
f \ast g = \int_0^t f(t-w)g(w) \, dw = \int_0^t (t-w)e^w \, dw
\]

\[
= t \int_0^t e^w - w e^w \, dw
\]

\[
= (te^w - we^w + e^w)|_0^t = e^t - t - 1
\]

\( \Box \)

The following result shows why convolutions are important:

<table>
<thead>
<tr>
<th>Laplace Transform of the Convolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ( f(t) ) and ( g(t) ) be two functions with Laplace transforms ( F(s) ) and ( G(s) ), respectively. Then:</td>
</tr>
<tr>
<td>( \mathcal{L}[f \ast g] = F(s)G(s) )</td>
</tr>
<tr>
<td>and</td>
</tr>
<tr>
<td>( \mathcal{L}^{-1}[F(s)G(s)] = f \ast g )</td>
</tr>
</tbody>
</table>

Example 5.31 Find \( \mathcal{L}[t^2 \ast e^t] \)
Solution: Since \( \mathcal{L}[f \ast g] = F(s)G(s) \), we have

\[
\mathcal{L}[f \ast g] = \frac{2}{s^3} \frac{1}{s - 1}
\]

We can use convolution as an alternative to partial fractions as shown next.

Example 5.32 Solve \( y'' + y = e^{2t} \), \( y(0) = 0 \), \( y'(0) = 0 \).

Solution: After taking Laplace transform of both sides we get:

\[
(s^2 + 1)Y(s) = \frac{1}{s - 2}
\]

or

\[
Y(s) = \frac{1}{s^2 + 1} \frac{1}{s - 2}
\]

so setting \( F(s) = \frac{1}{s^2 + 1} \) and \( G(s) = \frac{1}{s - 2} \) we see that

\( Y(s) = F(s)G(s) \) so \( y(t) = f \ast g \) where \( f(t) = \sin t \) and \( g(t) = e^{2t} \).

The convolution is

\[
\int_0^t e^{2(t-w)} \sin w \, dw
\]

which is

\[
\int_0^t e^{2(t-w)} \sin w \, dw.
\]

This is the same thing as

\[
e^{2t} \int_0^t e^{-2w} \sin w \, dw.
\]

At this point, we could integrate by parts to get the solution, but we wish to introduce a slick trick to avoid integration by PARTS, since the integrand looks like the definition of a Laplace transform,
where \( s = 2 \), and since \( 1 - u(w - t) \) is equal to zero for \( w > t \) and is equal to one for \( w < t \) (we view \( t \) as fixed), we may rewrite

\[
e^{2t} \int_0^t e^{-2w} \sin w \, dw = e^{2t} \int_0^t [1 - u(w - t)] e^{-2w} \sin w \, dw + e^{2t} \int_t^\infty [1 - u(w - t)] e^{-2w} \sin w \, dw
\]

\[
= e^{2t} \int_0^\infty [1 - u(w - t)] e^{-2w} \sin w \, dw
\]

\[
= e^{2t} \mathcal{L} [(1 - u(w - t)) \sin w] \tag{2}
\]

(note \( s = 2 \) in the Laplace transform definition).

\[
e^{2t} \left( \mathcal{L} [\sin w](2) - \mathcal{L}[u(w - t) \sin w] (2) \right)
\]

\[
e^{2t} \left( \mathcal{L} [\sin w](2) - \mathcal{L}[u(w - t) \sin(w - t + t)] (2) \right)
\]

\[
e^{2t} \left( \mathcal{L} [u(w - t) \sin(w - t)](2) + \mathcal{L}[u(w - t) \cos(w - t)](2) \right)
\]

\[
e^{2t} \left( \frac{1}{2^2 + 1} - \mathcal{L} [u(w - t) \sin(w - t)](2) - \sin t \mathcal{L}[u(w - t) \cos(w - t)](2) \right)
\]

\[
e^{2t} \left( \frac{1}{5} - \cos t \mathcal{L}[u(w - t) \sin(w - t)](2) - \sin t \mathcal{L}[u(w - t) \cos(w - t)](2) \right)
\]

\[
e^{2t} \left( \frac{1}{5} - \cos t \left( e^{-2t} \frac{1}{2^2 + 1} \right) - \sin t \left( e^{-2t} \frac{2}{2^2 + 1} \right) \right)
\]

\[
= \frac{1}{5} e^{2t} - \frac{1}{5} \cos t - \frac{2}{5} \sin t
\]
CHAPTER 5. LAPLACE TRANSFORMS

Perhaps this was better done with PARTS, but we wanted to illustrate the power of the Laplace transform.

The advantage of convolution is that we can solve any spring mass system without actually having the forcing function, as illustrated in the next example.

**Example 5.33** Solve \(y'' + y = g(t), \ y(0) = 0, \ y'(0) = 0\) for any \(g(t)\).

**Solution:** After taking Laplace transform of both sides we get:

\[(s^2 + 1)Y(s) = G(s)\]

or

\[Y(s) = \frac{1}{s^2 + 1}G(s)\]

so setting \(F(s) = \frac{1}{s^2 + 1}\) and we see that

\[Y(s) = F(s)G(s)\]

so \(y(t) = f \ast g\) where \(f(t) = \sin t\).

The convolution (and hence the solution) is

\[y(t) = \int_0^t \sin(t - w)g(w) \, dw\]

By superposition, we obtain the following:

### Convolution and Second Order Linear with Constant Coefficients

Consider

\[ay'' + by' + cy = g(t), \ y(0) = 0, \ y'(0) = 0.\]

The solution is \(y(t) = f \ast g\) where \(F(s) = \frac{1}{as^2 + bs + c}\), which is called the **transfer function** and we call \(f(t)\) **impulse response function** for this second order DE.

By superposition, we obtain the following:
Convolution and Second Order Linear with Constant Coefficients

Consider

\[ ay'' + by' + cy = g(t), \quad y(0) = c_1, \quad y'(0) = c_2. \]

If we have the particular solution to the homogeneous \( y_{\text{homo part}}(t) \) that satisfied the initial conditions \( y(0) = c_1 \) and \( y'(0) = c_2 \) then

\[ y(t) = y_{\text{homo part}}(t) + f * g(t) \]

will solve the nonhomogeneous IVP.

---

**Exercises**

*In 1-5, find the convolution.*

1. \( t^2 * t^3 \)
2. \( e^t * e^{3t} \)
3. \( \cos t * 1 \)
4. \( \cos t * \sin t \)
5. \( u(t-1) * 1 \)
6. Use convolution to solve \( y'' + y = 2, \quad y(0) = 0, \quad y'(0) = 0 \)
7. Use convolution to solve \( y'' - 4y = t, \quad y(0) = 0, \quad y'(0) = 0 \)
8. For a fixed constant \( \beta \), use convolution to solve \( y'' + 4y = \sin \beta t, \quad y(0) = 0, \quad y'(0) = 0 \)
9. Use an integral approximation to estimate \( y(1) \) for \( y'' + 4y = e^{t^2}, \quad y(0) = 0, \quad y'(0) = 0 \)
10. Find the impulse response function for an overdamped spring mass system \( my'' + by' + ky = g(t) \)
11. Find the impulse response function for an underdamped spring mass system \( my'' + by' + ky = g(t) \)