3.1 Introduction

In this chapter, we study second order ODE. Usually in a calculus or physics class one studies the second order equation:

\[ \frac{d^2y}{dt^2} = -32 \frac{ft}{sec^2} \]

where \( y(t) \) is the height (in feet) of a free falling object at time \( t \). This second order ODE can be solved simply by integrating twice to obtain:

\[ y(t) = -16t^2 + C_1t + C_2 \]

where \( C_1 \) and \( C_2 \) are constants that are determined by initial conditions (such as initial height and initial velocity. In particular, it is typical that general solutions to a second order ODE will have two free constants.

In this chapter we will study second order DE that cannot simply be solved by integrating.
CHAPTER 3. SECOND ORDER ODE

3.2 Homogeneous Second Order with Constant Coefficients

The first type of second order DE that we will study are homogeneous second order linear ODE with constant coefficients which are differential equations that can be written in the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

for constants $a, b, c$ which can be written as

$$ay'' + by' + cy = 0$$

where the independent variable is unspecified.

Let us consider the differential equation

$$y'' + 2y' + 3y = 0 \quad (3.1)$$

It is quite clear that any polynomial will not solve this equation, since if $y$ is a degree $n$ polynomial then $y''$ and $y'$ will be polynomials with degree strictly less than $n$. Therefore, the degree $n$ term that appears in the $3y$ in the above ODE will never cancel with any terms from $y''$ or $2y'$, hence we will never obtain zero.

Thus, a reasonable expectation is to search for functions whose derivatives look reasonably similar to the original function in hopes of cancellations. This leads us to consider the class of exponential functions $y = e^{rt}$ for for a fixed constant $r$.

Plugging into the left-hand side of the differential equation (3.1), we obtain

$$r^2e^{rt} + 2re^{rt} + 3e^{rt}$$

which factors as

$$e^{rt}(r^2 + 2r + 3).$$

Since $e^{rt} > 0$ for all $t$, if we intend for this quantity to be equal to zero (for all $t$) we must have

$$r^2 + 2r + 3 = 0.$$

Solving for $r$, we obtain $r = -3$ and $r = 1$. In other words we have shown that $y_1 = e^{-3t}$ and $y_2 = e^t$ both solve differential equation (3.1).
3.2. **HOMOGENEOUS SECOND ORDER WITH CONSTANT COEFFICIENTS**

Since $y_1$ and $y_2$ both solve the DE (3.1), one might guess that

$$y = c_1 y_1 + c_2 y_2$$

will also solve the DE, where $c_1$ and $c_2$ are constant. This is indeed the case since:

$$y' = c_1 y'_1 + c_2 y'_2$$

and

$$y'' = c_1 y''_1 + c_2 y''_2.$$

Plugging into the left-hand side of the DE (and factoring) we obtain:

$$y'' + 2y' + 3y =$$

$$c_1(y'_1 + 2y'_1 + 3y_1) + c_2(y'_2 + 2y'_2 + 3y_2)$$

However, the items in the parentheses above are zero since both $y_1$ and $y_2$ solve DE (3.1).

### 3.2.1 Real Distinct Roots Case

What we have done in the above discussion will work in general. In particular:

<table>
<thead>
<tr>
<th>Homogeneous constant coefficients–real distinct roots case</th>
</tr>
</thead>
<tbody>
<tr>
<td>If the homogeneous second order linear DE with constant coefficients</td>
</tr>
<tr>
<td>$ay'' + by' + cy = 0$</td>
</tr>
<tr>
<td>has associated polynomial</td>
</tr>
<tr>
<td>$ar^2 + br + c = 0$</td>
</tr>
<tr>
<td>and suppose that this polynomial has two distinct real roots $r_1$ and $r_2$, then</td>
</tr>
<tr>
<td>the general solution of (3.2) is given by</td>
</tr>
<tr>
<td>$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$</td>
</tr>
</tbody>
</table>

Note: The associated polynomial to DE (3.2) is called the characteristic polynomial of (3.2).
Example 3.1 Solve the DE

\[ y'' - 5y' + 6y = 0 \]

**Solution:** We form the characteristic polynomial.

\[ r^2 - 5r + 6 = (r - 2)(r - 3) \]

which clearly has roots \( r_1 = 2 \) and \( r_2 = 3 \). Hence the general solution is

\[ y = c_1e^{2t} + c_2e^{3t} \]

Example 3.2 Solve the DE

\[ \frac{d^2y}{dx^2} - 4 \frac{dy}{dx} - y = 0 \]

**Solution:** We form the characteristic polynomial.

\[ r^2 - 4r - 1. \]

This polynomial does not factor easily so we use the quadratic formula:

\[ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5} \]

so \( r_1 = 2 + \sqrt{5} \) and \( r_2 = 2 - \sqrt{5} \). Thus the general solution is

\[ y = c_1e^{(2+\sqrt{5})x} + c_2e^{(2-\sqrt{5})x}. \]

Note that the independent variable was specified to be \( x \) from the DE.

Recall from algebra (or by staring at the quadratic formula) that a quadratic polynomial

\[ ar^2 + br + c \]

has two distinct real roots if, and only if

\[ b^2 - 4ac > 0. \]

We note that this method even works if one of the roots of the quadratic formula is itself zero as the next example shows:
3.2. HOMOGENEOUS SECOND ORDER WITH CONSTANT COEFFICIENTS

Example 3.3 Solve the DE/IVP
\[ \frac{d^2z}{dt^2} - 3 \frac{dz}{dt} = 0, \quad z(0) = 1, \quad z'(0) = 2 \]

Solution: We form the characteristic polynomial.
\[ r^2 - 3r = r(r - 3) \]
which clearly has roots \( r_1 = 0 \) and \( r_2 = 3 \). Thus the general solution is
\[ z = c_1 e^{0t} + c_2 e^{3t} \]
or
\[ z = c_1 + c_2 e^{3t}. \]
Next, we use the initial conditions to find the correct values of the constants:
\[ 1 = z(0) = c_1 + c_2 \]
and \( z' = 3c_2 e^{3t} \) so \( z'(0) = 2 \) implies that \( c_2 = \frac{2}{3} \) and hence (by \( c_1 + c_2 = 1 \))
we see that \( c_1 = \frac{1}{3} \). So the particular solution to the IVP is
\[ z = \frac{1}{3} + \frac{2}{3} e^{3t}. \]

3.2.2 Repeated Roots Case
As mentioned at the beginning of the chapter, one would expect a second order DE to have a general solution with two free constants. If equation
\[ ay'' + by' + cy = 0 \]
has a characteristic polynomial
\[ ar^2 + br + c \]
that has two repeated real roots (which happens exactly when \( b^2 - 4ac = 0 \)), then our method previous runs into problems.
For instance, if we consider
\[ y'' + 6y' + 9y = 0 \]
we obtain
\[ r^2 + 6r + 9 = (r + 3)^2 \]
which has repeated roots \( r_1 = -3 \) and \( r_2 = -3 \).

When we form the general solution we get
\[ y = c_1 e^{-3t} + c_2 e^{-3t} \]
but this can be written as
\[ y = (c_1 + c_2) e^{-3t} = K_1 e^{-3t}. \]

So the trouble here is that we actually only have one free constant (so we are missing solutions to this DE). It turns out that another solution to this DE is obtained by \( z = te^{-3t} \) (one could check by differentiating and using the product rule). In a later section, we will motivate where this other solution comes from, but one might arrive at it from judicious guessing.

In the general repeated roots case:

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>If the homogeneous second order linear DE with constant coefficients</td>
</tr>
<tr>
<td>[ ay'' + by' + cy = 0 ] (3.3)</td>
</tr>
<tr>
<td>has associated polynomial</td>
</tr>
<tr>
<td>[ ar^2 + br + c = 0 ]</td>
</tr>
<tr>
<td>and suppose that this polynomial has two repeated real roots ( r_1 = r_2 ) (which occurs exactly when ( b^2 - 4ac = 0 )). then the general solution of (3.3) is given by</td>
</tr>
<tr>
<td>[ y = c_1 e^{r_1 t} + c_2 t e^{r_1 t} ]</td>
</tr>
</tbody>
</table>

**Example 3.4** Solve the DE
\[ y'' + 10y' + 25y = 0 \]

**Solution:** We form the characteristic polynomial.
\[ r^2 + 10r + 25 = (r + 5)^2 \]
which clearly has repeated roots \( r_1 = -5 \) and \( r_2 = -5 \). Hence the general solution is
\[ y = c_1 e^{-5t} + c_2 te^{-5t} \]
3.2. HOMOGENEOUS SECOND ORDER WITH CONSTANT COEFFICIENTS

3.2.3 Complex Roots Case

Our last case is if the DE

\[ ay'' + by' + cy = 0 \]

has a characteristic polynomial

\[ ar^2 + br + c \]

that has complex roots (which happens exactly when \( b^2 - 4ac < 0 \)). One should recall from precalculus (or by staring long enough at the quadratic formula) that complex roots come in conjugate pairs, which means if \( \alpha + \beta i \) is a complex root, then so is \( \alpha - \beta i \).

In particular, if \( b^2 - 4ac < 0 \) then the complex roots are:

\[ r_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i \]

(this is because \( \sqrt{b^2 - 4ac} = \sqrt{-1(4ac - b^2)} = \sqrt{4ac - b^2} i \))

If we label \( r_{1,2} = \alpha \pm \beta i \) then we can obtain the general solution to the DE as described below:

Homogeneous constant coefficients–repeated roots case
If the homogeneous second order linear DE with constant coefficients

\[ ay'' + by' + cy = 0 \]  \hspace{1cm} (3.4)

has associated polynomial

\[ ar^2 + br + c = 0 \]

and suppose that this polynomial has complex roots \( r_{1,2} = \alpha \pm \beta i \) (which occurs exactly when \( b^2 - 4ac < 0 \)). then the general solution of (3.4) is given by

\[ y = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) \]

or

\[ y = c_1 e^{-\left(\frac{b}{2a}\right)t} \cos \left( \frac{\sqrt{4ac - b^2}}{2a} t \right) + c_2 e^{-\left(\frac{b}{2a}\right)t} \sin \left( \frac{\sqrt{4ac - b^2}}{2a} t \right) \]
Example 3.5 \textit{Solve the DE}
\begin{equation*}
y'' + 2y' + 2y = 0
\end{equation*}
\textbf{Solution:} We form the characteristic polynomial.
\begin{equation*}
r^2 + 2r + 2
\end{equation*}
which, by the quadratic formula has roots
\begin{equation*}
r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2}
\end{equation*}
\begin{equation*}
= \frac{-2 \pm 2i}{2} = -1 \pm i
\end{equation*}
so \( r_{1,2} = 1 \pm i \) so \( \alpha = -1 \) and \( \beta = 1 \). 
Hence the general solution is
\begin{equation*}
y = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t \quad \square
\end{equation*}

Example 3.6 \textit{Solve the DE}
\begin{equation*}
y'' + 4y = 0
\end{equation*}
\textbf{Solution:} We form the characteristic polynomial.
\begin{equation*}
r^2 + 4
\end{equation*}
which, by the quadratic formula has roots \( r_{1,2} = \pm 2i \) so \( \alpha = 0 \) and \( \beta = 2 \). 
Hence the general solution is
\begin{equation*}
y = c_1 \cos(2t) + c_2 \sin(2t) \quad \square
\end{equation*}

\textbf{Exercises}

\textit{Find general solutions for each of the DEs, note that the roots are real}

\begin{enumerate}
\item \( y'' + 4y' + 4y = 0 \)
\item \( y'' + 3y' + 2y = 0 \)
\end{enumerate}
3.2. **HOMOGENEOUS SECOND ORDER WITH CONSTANT COEFFICIENTS**

3. $y'' + 6y' - 7y = 0$

4. $z'' + 4z' + z = 0$

5. $z'' - z = 0$

6. $z'' + 2z' = 0$

*Find general solutions for each of the DEs*

7. $y'' + 4y' + 6y = 0$

8. $y'' + 6y' + 10y = 0$

9. $y'' - 7y = 0$

10. $y'' + 7y = 0$

11. $z'' + z' + z = 0$

12. $z'' + 2z' + 5z = 0$

*Find particular solutions for each of the DE/IVP*

13. $y'' + 4y' + 3y = 0$, $y(0) = 1$, $y'(0) = -1$

14. $y'' + 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$

15. $y'' + 8y' + 17y = 0$, $y(0) = 0$, $y'(0) = 1$

16. $y'' - 9y' + 20y = 0$, $y(0) = 4$, $y'(0) = -2$

17. $y'' + 16y = 0$, $y(0) = 4$, $y'(0) = -2$

18. $z'' - 6z = 0$, $z(0) = 1$, $z'(0) = 2$

19. $z'' - 6z = 0$, $z(0) = 0$, $z'(0) = 0$