2.4 Applications (Exponential Growth/Decay)

Many things grow (decay) as fast as exponential functions. In general, if a quantity grows or decays at a rate proportional to quantity itself, then it will exhibit exponential behavior.

\[
\frac{dQ}{dt} = rQ
\]

(\(r\) is the constant of proportionality) Then

\[
Q(t) = Ke^{rt}, \quad \text{with } K = Q(0)
\]

Proof: Clearly if \(Q(t) = Ke^{rt}\), then

\[
Q'(t) = rKe^{rt} = rQ(t).
\]

Plugging \(t = 0\) into \(Q(t) = Ke^{rt}\), we see that \(Q(0) = K\).

Note in equation (2.4) that \(r\) is determined by the rate at which the quantity grows or decays. Clearly this depends upon the units with which \(Q\) is measured and with which time is measured.

2.4.1 Radioactive Decay

It is well-known that radioactive materials decay at a rate proportional to the amount of material present.

Example 2.9 (Carbon Dating) Carbon-14 has a half-life of 5,730 years. A fossil is found to have 10\% of its original Carbon-14. Determine the age of the fossil.

Solution: Let \(Q(t)\) be the amount of Carbon-14 present in the fossil at time \(t\), where \(t = 0\) is the time of the fossilized animal. The half-life is clearly
related to the quantity $r$, since both concern the rate of decay. So we first determine $r$ by noticing that when $t = 5730$ there will be one-half of the original amount of Carbon-14, or

$$Q(5730) = \frac{1}{2}Q(0).$$

In particular

$$Q(0)e^{5730r} = \frac{1}{2}Q(0),$$

so

$$e^{5730r} = \frac{1}{2},$$

$$r = \frac{\ln(\frac{1}{2})}{5730} = -\frac{\ln 2}{5730}. $$

Next, we solve for the value of $t$ that yields 10% of its original Carbon-14. That is we wish to solve

$$Q(t) = .1Q(0)$$

for $t$.

$$Q(0)e^{-\frac{\ln 2}{5730}t} = \frac{1}{10}Q(0),$$

$$e^{-\frac{\ln 2}{5730}t} = \frac{1}{10},$$

$$-\frac{\ln 2}{5730}t = \ln(\frac{1}{10}),$$

$$t = \frac{5730\ln 10}{\ln 2} \approx 19034.648 \text{ years}.$$ 

### 2.4.2 Population Models

A biological population that is not subject to resource limitations will grow at a rate proportional to itself. This is entirely believable, since if the size of the population increases by a factor of $k$ then we would expect the growth rate to also increase by a factor of $k$ (roughly, the number of babies doubles if the population doubles). Hence, if $P(t)$ the population of a specific species at time $t$, then it is reasonable that \( \frac{dP}{dt} = rP \). This is called a Malthusian
or exponential population model. Such populations grow exponentially, and this model works reasonably well when the population in question is not close to being constrained by a lack of resources. Clearly, if \( r > 0 \) the population is growing and if \( r < 0 \) the population is dying.

**Example 2.10 (Bacteria Counts)** Escherichia coli (E. Coli) is measured in colony forming units per milliliter (CFU/mL). In 'ideal' circumstances E. Coli in dairy milk has a doubling time of roughly 20 minutes. Pasteurized milk is Grade A if it has less than 1,000 CFU/mL. How long will it take for a gallon of milk with 1,000 CFU/mL to reach 1,000,000 CFU/mL (which is considered harmful) when left in 'ideal' circumstances?

**Solution:** Let \( Q(t) \) be the CFU/mL of E. Coli at time \( t \) (minutes). The doubling rate determines \( r \) (just as half-life determines \( r \)).

In particular
\[
Q(20) = 2Q(0)
\]
so
\[
Q(0)e^{20r} = 2Q(0)
\]
or
\[
r = \frac{1}{20} \ln 2.
\]

We wish to solve for \( t \) in
\[
Q(t) = 10^6,
\]
where \( Q(0) = 10^3 \). Solving, we obtain
\[
10^3e^{\frac{1}{20} \ln 2 t} = 10^6
\]
so
\[
t = 20 \cdot \frac{\ln 10^3}{\ln 2} \approx 199.32 \text{ minutes}
\]
or about 3 hours and 19.32 minutes. \( \square \)
2.4. APPLICATIONS (EXPONENTIAL GROWTH/DECAY)

2.4.3 Financial Models

Money invested at interest generally grows in proportion to the amount of money that is invested (principal).

Example 2.11 (Saving for Retirement) A 25 year-old has inherited $50,000.00 and plans to invest it in an investment that pays 5% annual interest for 40 years. How much will be in the bank after 40 years?

Solution: Let $Q(t)$ be the dollar value of the investment at time $t$ (years). Then $\frac{dQ}{dt} = rQ$ and $Q(0) = 50,000$.

It is somewhat clear that the interest rate will dictate $r$, but how? After 1 year, we expect a growth of 5% so

$$Q(1) = Q(0) + (.05)Q(0)$$

or

$$Q(0)e^r = 1.05Q(0).$$

So,

$$r = \ln 1.05 \approx 0.04879016417$$

So, to solve the problem, we want

$$Q(40) = 50000e^{40\ln 1.05} = 50000 \cdot (1.05)^{40} = $351,999.44 \Box$$

One should always realize a mathematical model does not represent the actual quantities precisely. In truth, bacteria and investments do not grow continuously. Moreover, the actual quantities involved in population models can only be integers and the amount of dollars is at best measured to two decimal places. However, these models do an excellent job reflecting the real quantities that that they model.

Exercises

Use the fact that $C^{14}$ has a half life of around 5730 years.

1. How old is a fossil with 25% of its original $C^{14}$?

2. Suppose that a researcher is confident that a fossil has somewhere between 15 – 35% of its original $C^{14}$. What is the range of possible ages of this fossil?
3. The production of the first atomic bomb also produced the byproduct of 1881 Ci (Curie) of the radioactive isotope Radium-226 which has a half life of 1600 years. This material is currently stored in Lewiston, NY (my hometown!!). Even 1 Curie of this material is extremely hazardous, but compute how long it will take for the 1881 Ci of Ra\(^{226}\) to decay to 1 Curie.

4. A rabbit population doubles every 6 months. If the colony starts with 500 rabbits, how long will it take to reach 1500?

5. (a) An $5000 investment is made for 30 years at 8% annual interest. How much will the investment be worth?

(b) Additionally, the investor plans to add $1200 each year to his investment. How much will the investor have after 40 years? [Hint: use the (linear) DE \( \frac{dQ}{dt} = K + rQ \), where \( r \) is determined by the interest rate, and \( K \) is determined \( Q(1) = 6600 \).]

6. A loan of $100,000 is taken out at 5% annual interest for 30 years.

(a) Assume it is paid off at a continuous (and constant) rate \( K \). Determine \( K \) so that the loan is completely paid off in 30 years. [Hint: use the (linear) DE \( \frac{dQ}{dt} = rQ - K \), where \( r \) is determined by the interest rate, and \( K \) is determined \( Q(30) = 0 \).]

(b) What annual payment does this value of \( K \) correspond to?
2.5 More Applications

2.5.1 Newton’s Law of Cooling

Newton’s Law of Cooling (see Example 1.6) states the rate of change of the temperature of an object is proportional to the difference of the temperature $T$ of that object and the ambient temperature $M$ of the room. If $T(t)$ denotes the temperature of the object at time $t$, then this translates to

$$\frac{dT}{dt} = k(T - M),$$

where $k$ is a constant that determined by the cooling rate. This is a first order linear ODE, which can easily be solved.

Example 2.12 (Forensics) A corpse is found at noon in a room that is held at a constant temperature of 70°F. The body is determined to have a temperature of 78°F. One half hour later, the body cools to 76°F. Assuming that the deceased person had a body temperature of 98.6°F at the time of death, determine the time of death to the nearest minute.

Solution: Let $T(t)$ denotes the temperature of the object at time $t$ (in hours), where $t = 0$ is noon. By Newton’s Law of Cooling

$$\frac{dT}{dt} = k(T - 70)$$

or

$$\frac{dT}{dt} - kT = -70k.$$

Using the first order linear formula:

$$T(t) = \frac{-70ke^{-kt}}{e^{-kt}} + C$$

$$T(t) = (70e^{-kt} + C)e^{kt} = 70 + Ce^{kt}$$

We recover $C$ by using the information $T(0) = 78$, which gives $C = 8$

Next, we recover $K$ by using the information $T(\frac{1}{2}) = 76$

$$76 = 70 + 8e^{\frac{1}{2}k},$$
So
\[ k = 2 \ln \left( \frac{6}{8} \right) = \ln \left( \frac{9}{16} \right). \]

Finally, we solve for \( T(t) = 98.6 \) for \( t \) (expecting a negative value of \( t \)).

\[ 98.6 = 70 + 8e^{\ln\left(\frac{2}{16}\right)t} = 70 + 8 \left( \frac{9}{16} \right)^t \]

So
\[ \frac{28.6}{8} = \left( \frac{9}{16} \right)^t \]

\[ t = \frac{\ln(28.6) - \ln 8}{\ln 9 - \ln 16} \approx -2.214189 \text{ hours} \]

So the person died approximately 2 hours and 13 minutes ago which would have been about 9:47 AM.

Newton’s Law of Cooling works equally well in a heating situation, where the initial temperature of the object is below the ambient temperature.

2.5.2 Mixture Problems

Differential equations also lend themselves to mixing problems as in the following example. We provide a few tips for writing the necessary differential equation.
Tips for Setting Up a Model Using a Differential Equation

1. Define the unknown function so that it solves the problem at hand, in almost all examples, quantities should be absolute quantities (avoid relative quantities, e.g., percentages). Clearly and specifically define the units of both dependent and independent variables.

2. Write a differential equation based on the rates at which the desired quantity is changing (a sketch may help). In general, avoid using initial data information, as this will be used for the initial condition. Recall that the rate of change of the quantity in question can often be thought of as

\[
\text{rate of increase } - \text{ rate of decrease}
\]

3. Write the initial condition.

Example 2.13 (A Mixture Problem) A 200 liter vat initially contains a 10% solution of Hydrogen Peroxide and 90% water. If a 1% hydrogen peroxide-water solution is added to the vat at a constant rate of 2L/min and the mixture is drained off at the same rate, determine how long it will take until the mixture reaches a concentration of 3% Hydrogen Peroxide. The vat is well-stirred, so assume that the Hydrogen Peroxide is uniformly distributed in the vat.

Solution: Let \(M(t)\) denote the liters of hydrogen peroxide in the tank at time \(t\), where \(t\) is measured in minutes.

Since the mixture entering the vat at a rate of 2L/min is 1% hydrogen peroxide, the rate of hydrogen peroxide is entering the tank is \(0.02\)L/min.

The rate of mixture exiting the tank is also 2L/min. However, not all of this is hydrogen peroxide. With the assumption that the mixture is uniformly mixed, for any time \(t\), we know that the amount of hydrogen peroxide in 1L of the mixture in the vat is \(\frac{M(t)}{200}\). Thus

\[
\frac{dM}{dt} = \text{rate of increase of } M - \text{rate of decrease of } M = 0.02 - 2 \frac{1}{200} M.
\]

The initial condition is \(M(0) = 20\).
CHAPTER 2. METHODS FOR SOLVING FIRST ORDER ODES

The DE is a first order linear
\[
\frac{dM}{dt} + \frac{1}{100}M = \frac{2}{100}
\]
Solving:
\[
M(t) = \int \frac{2}{100}e^{\frac{t}{100}} dt + C
\]
\[
= \left(2e^{\frac{t}{100}} + C\right) \cdot e^{-\frac{t}{100}} = 2 + Ce^{-\frac{t}{100}}
\]
As usual, the initial condition gives the particular value of \(C\), namely \(M(0) = 20\) implies that \(C = 18\), so
\[
M(t) = 2 + 18e^{-\frac{t}{100}}
\]
We want to know at what \(t\) will the solution reach a 3% concentration of hydrogen peroxide. Hence, we seek to solve
\[
\frac{M(t)}{200} = .03
\]
or
\[
M(t) = 6.
\]
So we are solving
\[
6 = 2 + 18e^{-\frac{t}{100}}
\]
for \(t\).
We obtain
\[
4 = 18e^{-\frac{t}{100}}
\]
or
\[
\frac{4}{18} = e^{-\frac{t}{100}},
\]
so
\[
t = -100\ln\left(\frac{2}{9}\right) \approx 150.40774 \text{ minutes}
\]
or about \(2\frac{1}{2}\) hours.

We solve another mixture type problem.
2.5. MORE APPLICATIONS

Example 2.14 (Another Mixture Problem) A 50 gallon storage tank initially holds a saltwater solution containing 100 grams of salt dissolved in 50 gallons of water. A salt/water mixture is pumped in at a rate of 2 gallon per minute. The incoming mixture has 4 grams of salt per gallon of mixture. The salt water is also pumped out at a rate of 3 gallons per minute. How much salt is in the tank when the tank is half full? What is the concentration of salt (grams/gallon) in the mixture that is exiting the tank as it becomes empty?

Solution: Let \( M(t) \) denote the amount of salt (in grams) in the tank at time \( t \), where \( t \) is measured in minutes.

Since the mixture entering the tank at a rate of 2 gallons/min and each gallon contains 4 gram of salt, there are 8 grams of salt entering per minute.

The rate of mixture exiting the tank is 3 gallons/min. Assuming that the salt is evenly dispersed in the tank, the amount of salt in one gallon of mixture in the tank is given by \( \frac{M(t)}{50 - t} \) where \( 50 - t \) is the total amount (in gallons) of solution in the tank. So in all, the rate of salt exiting the tank is \( \frac{3M}{50 - t} \).

Thus

\[
\frac{dM}{dt} = \text{rate of increase of } M - \text{rate of decrease of } M = 8 - \frac{3}{50 - t}M.
\]

Or

\[
\frac{dM}{dt} + \frac{3}{50 - t}M = 8.
\]

Solving

\[
M(t) = \int \frac{8e^{-3 \ln(50-t)}}{e^{-3 \ln(50-t)}} dt + C
\]

\[
M(t) = \int \frac{8}{(50-t)^3} dt + C
\]

\[
M(t) = \left( \frac{4}{(50-t)^2} + C \right)(50 - t)^3 = 4(50 - t) + C(50 - t)^3
\]

As usual, the initial condition gives the particular value of \( C \), namely \( M(0) = 100 \) implies that 100 = 200 + 503C.
\[ C = \frac{-100}{50^3} \]

so

\[ M(t) = 4(50 - t) - \frac{100}{50^3}(50 - t)^3 \]

Since the tank will be half full when \( t = 25 \) we have

\[ M(25) = 4(25) - \frac{100}{50^3}25^3 = 100 - 100(\frac{1}{2})^3 = 100 + 12.5 = 87.5 \text{ grams} \]

The number of grams of salt per gallon at any time \( t \) is given by \[ \frac{M(t)}{50 - t} = 4 - \frac{100}{50^3}(50 - t)^2 \]
so as the tank empties, \( t = 50 \) so the solution has a concentration of 4 grams of salt per gallon.

\[ \square \]

**Exercises**

Assume that living persons have a body temperature of 98.6° F.

1. Mr. Boddy’s body is discovered in a walk-in freezer that is held at 34° F. His body has a temperature of 42° F. An hour later, his body cools to 38° F. Find how long ago Mr. Boddy died from the initial time that he was discovered.

2. How long will it take a 40° F glass of milk to heat to an undrinkable 60° F in a room that is held at 72° F? (Assume that after 1 minute, the milk increases from 40° F to 43° F).

3. A 100 gallon tank holds 50 gallons of water with 75 grams of salt dissolved into it. If a mixture with 6 grams per gallon of salt is pumped in at 2 gallons per hour and the mixture is drained at 1 gallon per hour, determine how much salt will be in the tank when it is full.
2.6 Population Model-Logistic Model

We have already investigated population models where the population is not limited by resources. As in Equation (2.4), the population behaves as an exponential function which is unbounded. In this section, we seek to create a model that takes resource limitations into account. We start with a concrete example and then present the general logistic model.

**Example 2.15 (A rabbit colony)** A colony of 1000 rabbits lives in a field that can support 5000 rabbits. Write a differential equation that takes the resource limitation into account.

**Solution:** Let $P(t)$ be the population at time $t$. As described previously, in the absence of resource limitations, a good model is:

$$\frac{dP}{dt} = rP.$$

We want

$$\frac{dP}{dt} > 0 \quad \text{when } 0 < P < 5000$$

and

$$\frac{dP}{dt} < 0 \quad \text{when } P > 5000.$$

This implies that we want

$$\frac{dP}{dt} = 0 \quad \text{when } P = 5000.$$

A simple way to achieve this is allow $r$ in Equation (2.4) to be a function of $P$, and make $r$ negative if $P > 5000$. We do this by letting $r = k(5000 - P)$, and we obtain the DE

$$\frac{dP}{dt} = k(5000 - P)P$$

where $k$ is a constant that is determined by the growth rate of the population. □

In the previous example, the population was limited by resources to 5000 rabbits. This value is called a carrying capacity or limiting population. In general,
The Logistic Differential Equation

A population \( P \) at time \( t \) with a carrying capacity of \( P_\infty \) is modeled by the logistic differential equation (or logistic growth model)

\[
\frac{dP}{dt} = kP(P_\infty - P)
\]

where \( k > 0 \) is a constant that is determined by the growth rate of the population.

Note: It is somewhat standard to write the logistic differential equation as

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{P_\infty}\right)
\]

This calibrates the value of \( k \), making it comparable with the growth rate \( r \) in an exponential model, which can be useful in specific applications.

Next, we solve the separable differential equation (2.6) as follows:

\[
\frac{dP}{P(P_\infty - P)} = k \, dt
\]

(with \( P(t) \neq 0 \) and \( P(t) \neq P_\infty \), which are two constant solutions).

\[
\int \frac{dP}{P(P_\infty - P)} = \int k \, dt
\]

Using partial fractions on the left integrand:

\[
\int \frac{1}{P} - \frac{1}{P_\infty - P} \, dP = \int k \, dt
\]

\[
\frac{1}{P_\infty} \int \frac{1}{P} - \frac{1}{P_\infty - P} \, dP = \int k \, dt
\]

\[
\frac{1}{P_\infty} \left(\ln |P| - \ln |P_\infty - P|\right) = kt + C
\]

\[
\ln |P| - \ln |P_\infty - P| = P_\infty(kt + C)
\]

\[
\ln \left|\frac{P}{P_\infty - P}\right| = P_\infty(kt + C)
\]
2.6. POPULATION MODEL-LOGISTIC MODEL

\[ \frac{P}{P_{\infty} - P} = e^{P_{\infty}(kt + C)} \]

Note that we can let \( \tilde{C} = e^{P_{\infty}C} \)

\[ \frac{P}{P_{\infty} - P} = \tilde{C}e^{P_{\infty}kt} \]

where \( \tilde{C} > 0 \). Dropping absolute value on the left amounts to dropping the positivity condition on \( C \) so

\[ \frac{P}{P_{\infty} - P} = \tilde{C}e^{P_{\infty}kt} \]

or

\[ P = (P_{\infty} - P)\tilde{C}e^{P_{\infty}kt} \]

\[ P + P\tilde{C}e^{P_{\infty}kt} = P_{\infty}\tilde{C}e^{P_{\infty}kt} \]

\[ P = \frac{P_{\infty}\tilde{C}e^{P_{\infty}kt}}{1 + \tilde{C}e^{P_{\infty}kt}} \]

Multiplying by \( \frac{e^{-P_{\infty}kt}}{e^{-P_{\infty}kt}} \) we obtain

\[ P(t) = \frac{P_{\infty}\tilde{C}}{\tilde{C} + e^{-P_{\infty}kt}} \]

We can solve for \( \tilde{C} \) in terms of \( P(0) \), the initial population,

\[ P(0) = \frac{P_{\infty}\tilde{C}}{\tilde{C} + 1} \]

So

\[ P(0)(\tilde{C} + 1) = P_{\infty}\tilde{C} \]

giving

\[ \tilde{C} = \frac{P(0)}{P_{\infty} - P(0)} \]

or

\[ P(t) = \frac{P_{\infty}P(0)}{P_{\infty} - P(0) + e^{-P_{\infty}kt}} \]
which simplifies to

\[ P(t) = \frac{P_{\infty}P(0)}{P(0) + [P_{\infty} - P(0)] e^{-P_{\infty}kt}} \]

Note that the constant solution \( P(t) = 0 \) is obtained if \( P(0) = 0 \), and the constant solution \( P(t) = P_{\infty} \) is obtained if \( P(0) = P_{\infty} \).

We have proven that

<table>
<thead>
<tr>
<th>Solutions of the Logistic Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>The solutions to ( \frac{dP}{dt} = kP(P_{\infty} - P) )</td>
</tr>
<tr>
<td>are given by ( P(t) = \frac{P_{\infty}P_0}{P_0 + [P_{\infty} - P_0] e^{-P_{\infty}kt}} )</td>
</tr>
<tr>
<td>where ( P_0 ) is ( P(0) ) and ( P_{\infty} ) is the limiting population (carrying capacity).</td>
</tr>
</tbody>
</table>

One should note that as \( t \) approaches infinity, \( P(t) \) limits to the limiting population \( P_{\infty} \) or

\[
\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{P_{\infty}P_0}{P_0 + [P_{\infty} - P_0] e^{-P_{\infty}kt}} = P_{\infty}.
\]

**Example 2.16 (A rabbit colony-continued)** A colony of 1000 rabbits lives in a field that can support 5000 rabbits. After 1 month, the colony reaches 1050 rabbits. Use a logistic model to predict when the population will have 2500 rabbits.

**Solution:** Let \( P(t) \) be the population of rabbits at time \( t \) (in months). By Equation (2.6)

\[ P(t) = \frac{P_{\infty}P_0}{P_0 + [P_{\infty} - P_0] e^{-P_{\infty}kt}} \]

where \( P_0 = 1000 \) and \( P_{\infty} = 5000 \). So

\[
P(t) = \frac{1000 \cdot 5000}{1000 + 4000e^{-5000kt}} = \frac{5000}{1 + 4e^{-5000kt}}
\]
We need to compute $k$ using the information that $P(1) = 1050$. So

$$1050 = P(1) = \frac{5000}{1 + 4e^{-5000k}}$$

which solving for $k$ gives

$$k = \frac{-1}{5000} \ln \left( \frac{1}{4} \left( \frac{5000}{1050} - 1 \right) \right) = \frac{-1}{5000} \ln \left( \frac{79}{84} \right)$$

So

$$P(t) = \frac{5000}{1 + 4e^{\ln \left[ \frac{79}{84} \right] t}}$$

We solve for $P(t) = 2500$ and obtain

$$2500 = \frac{5000}{1 + 4 \left[ \frac{79}{84} \right]^t}$$

$$1 + 4 \left[ \frac{79}{84} \right]^t = 2$$

$$4 \left[ \frac{79}{84} \right]^t = 1$$

$$\left[ \frac{79}{84} \right]^t = \frac{1}{4}$$

$$t = \ln \left( \frac{1}{4} \right) \frac{1}{\ln \left[ \frac{79}{84} \right]} \approx 22.589 \text{ months} \square$$

A Note: in the previous problem, had we not cancelled (especially inside the exponential), we would have introduced serious round off error in the computation and our calculation would have been far off.

**Exercises**

1. The world’s (human) population went from 4 billion in 1975 to six billion in 2000. If the carrying capacity is 20 billion, estimate at what year the human population will reach 19 billion (and will start to be limited by resources). [Hint: measure $P$ in billions, i.e. $P(0) = 4$].
2. A certain ant colony has grown from 5000 to 6000 ants in 6 months. The colony has a carrying capacity of 10000. Find how long it will take until the colony reaches a size of 9000.

3. Prove (using calculus) that the change of concavity of all solutions of Equation (2.6) with \(0 < P_0 < P_\infty\) take place when \(P = \frac{1}{2}P_\infty\). Explain how this can be used to estimate a population’s carrying capacity given only a graph of the population over time.

2.7 Geometric Interpretation, Slope Fields, and Euler’s Method

Any first order differential equation can be thought of geometrically. In particular the DE
\[
\frac{dy}{dx} = f(x, y)
\]
specifies a specific slope for any point \((x, y)\) given by the function \(f\).

A **slope field** is simply a plot of the slope
\[
\frac{dy}{dx} = f(x, y)
\]
for several points \((x, y)\) represented by a vector having slope \(f(x, y)\) based at \((x, y)\), usually plotted with a fixed change in \(x\) (or so that all vectors have a uniform length).

**Example 2.17** For \(\frac{dy}{dx} = \frac{1}{2}(x^2 - y)\), plot the associated vectors at the points: \((1, 2), (1, 3), (2, 2), (2, 3)\), where the vectors are pictured having a change in \(x\) of 1 \((\Delta x = 1)\).

**Solution:**

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(\frac{dy}{dx})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 2))</td>
<td>(-\frac{1}{2})</td>
</tr>
<tr>
<td>((1, 3))</td>
<td>(-1)</td>
</tr>
<tr>
<td>((2, 2))</td>
<td>(1)</td>
</tr>
<tr>
<td>((2, 3))</td>
<td>(\frac{1}{2})</td>
</tr>
</tbody>
</table>