2.2. FIRST ORDER LINEAR ODE

2.2.1 Solving First Order Linear ODE

A differential equation is called a *linear* first order ODE if it can be rewritten into the form:

\[ a_1(x) \frac{dy}{dx} + a_0(x)y = g(x). \]  \hspace{1cm} (2.2)

Here \( a_1(x) \), \( a_0(x) \) and \( g(x) \) are functions of \( x \). So long as \( a_1(x) \neq 0 \), by dividing, we can write any first order linear differential equation in *standard form*, shown below.

<table>
<thead>
<tr>
<th>Standard Form of a First Order Linear ODE</th>
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<tbody>
<tr>
<td>A first order linear ODE is said to be in standard form if it is in the form</td>
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<tr>
<td>[ \frac{dy}{dx} + P(x)y = Q(x) ]</td>
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<tr>
<td>(2.3)</td>
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<tr>
<td>for functions ( P(x) ) and ( Q(x) ).</td>
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Any first order linear has a solution given below:

<table>
<thead>
<tr>
<th>Solving a First Order Linear ODE in Standard Form</th>
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<tr>
<td>Consider the differential equation</td>
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<tr>
<td>[ \frac{dy}{dx} + P(x)y = Q(x). ]</td>
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<tr>
<td>If ( P(x) ) has antiderivative ( \int P )</td>
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<tr>
<td>[ y = \frac{\int Q(x)e^{\int P} , dx + C}{e^{\int P}}. ]</td>
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<tr>
<td>(Here, ( C ) is the constant of integration of the outer integral in the numerator).</td>
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*Proof:* Use the quotient rule to differentiate

\[ y = \frac{\int Q(x)e^{\int P} \, dx + C}{e^{\int P}}. \]
with respect to $x$ and the fact that $\frac{d}{dx} \int P = P(x)$ obtain

$$\frac{dy}{dx} = \frac{(Q(x)e^{\int P})e^{\int P} + (\int Q(x)e^{\int P} dx + C)e^{\int P}P(x)}{(e^{\int P})^2}$$

The right hand side simplifies to

$$= Q(x) - P(x)\left(\frac{\int Q(x)e^{\int P} dx + C}{e^{\int P}}\right) = Q(x) - P(x)y \qed$$

**Example 2.4** Solve the DE

$$\frac{dz}{dx} = 2z + x; \ x > 0$$

**Solution:** We first write the DE in standard form:

$$\frac{dz}{dx} - \frac{2}{x}z = x.$$ 

We then identify $P(x) = -\frac{2}{x}$ and $Q(x) = x$. Next, compute $\int P = -2\ln x$ (note that only one antiderivative is required and since $x > 0$, we do not write $\ln |x|$).

Next we compute $e^{\int P}$:

$$e^{\int P} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}.$$ 

Substituting into the formula:

$$y = \frac{\int Q(x)e^{\int P} dx + C}{e^{\int P}},$$

we obtain

$$y = \frac{\int x \cdot \frac{1}{x^2} dx + C}{\frac{1}{x^2}}$$

and finally get

$$y = \frac{\ln x + C}{\frac{1}{x^2}} = x^2 \ln x + Cx^2. \qed$$

For a first order linear differential equation in standard form, the expression $e^{\int P}$ is called the integrating factor. Instead of simply memorizing the
above formula, an alternate way to solve equations of the form (2.2.1) is to multiply equation (2.2.1) by the integrating factor $\exp \int P$ and then realizing that the resulting left-hand side is equal to $\frac{d}{dx} (y \cdot \exp \int P)$ by the product rule.

**Example 2.5** Solve the DE

$$\frac{dy}{dx} = x - 3y$$

using the alternate method described above.

**Solution:** We first write the DE in standard form:

$$\frac{dy}{dx} + 3y = x.$$ 

We then identify $P(x) = 3$ and $Q(x) = x$. Multiplying both sides of the differential equation by the integrating factor $e^{\int P} = e^{3x}$ we obtain:

$$e^{3x} \frac{dy}{dx} + 3e^{3x} y = xe^{3x}.$$ 

Note that by the product rule (and chain rule), the left-hand side of the above expression is simply $\frac{d}{dx} (e^{3x} y)$

Rewriting, we obtain

$$\frac{d}{dx} (e^{3x} y) = xe^{3x}.$$ 

We integrate both sides with respect to $x$ and obtain

$$e^{3x} y = \int xe^{3x} \, dx$$

The right-hand side (after integration by Parts) simplifies to

$$\frac{1}{3} xe^{3x} - \frac{1}{3} e^{3x} + C$$

Hence,

$$e^{3x} y = \frac{1}{3} xe^{3x} - \frac{1}{3} e^{3x} + C$$
and solving for $y$, we obtain

$$y = \frac{1}{3}x - \frac{1}{3} + Ce^{-3x}$$

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**Exercises**

*Solve the following DE:*

1. \( \frac{dy}{dx} - \frac{1}{x}y = x^3; \quad x > 0 \)
2. \( \frac{dy}{dx} - y = e^{2x} \)
3. \( \frac{dz}{dt} = t + z \)
4. \( x \frac{dy}{dx} = (\sin x - y) \, dx \)
5. \( \frac{dz}{dt} = \cos t - z \cot t; \quad 0 < t < \frac{\pi}{2} \)
6. \( \frac{dy}{dx} = \frac{y}{2y - x} \) (Hint: solve for $x$ in terms of $y$)

*Solve the following initial Value Problems*

7. \( \frac{dy}{dx} + \frac{3}{x}y = 1; \quad y(1) = 2 \)
8. \( \frac{dy}{dx} + y = 2; \quad y(0) = -1 \)
9. \( \frac{dz}{dx} + 2xz = x; \quad y(1) = 2 \)