3.3 Second Order with Constant Coefficients-
Undetermined Coefficients

In the previous section we learned how to solve all possible homogeneous second order linear ODE with constant coefficients, written as:

\[ ay'' + by' + cy = 0 \]

for constants \( a, b, c \).

In this section we will solve

\[ ay'' + by' + cy = f(t) \]

for a specific class of functions \( f(t) \).

3.3.1 Finding one solution to the nonhomogeneous

We motivate this method with an example:

\[ y'' + 5y' + 6y = 4e^t \]

Our goal is to find one solution to this differential equation. In particular, we ask ourselves, what kind of functions might possibly solve this nonhomogeneous differential equation? Clearly, polynomials or trigonometric functions will not do. The only possible solution would be a function that involves \( e^t \) so we make a reasonable guess at the form of a solution by considering functions of the form \( y = Ae^t \). Our strategy will be to plug this function into the DE and solve for the constant \( A \) to force the function to solve the differential equation.

After plugging this function into the left side of DE we obtain

\[ y'' + 5y' + 6y = Ae^t + 5Ae^t + 6Ae^t = 12Ae^t \]

So in order to solve the DE (to match the right hand side) we must choose \( 12A = 4 \), or \( A = \frac{1}{3} \) in order to obtain \( 4e^t \). Thus, we have found that \( y = \frac{1}{3}e^t \) solves the DE.

We seem to have been extremely lucky in the previous example. Since the function \( y = Ae^t \) has derivatives that have the same form as the function itself. If the function \( f(t) \) had been \( 4 \cos(2t) \) then we would have made a guess of the form \( y = A \cos(2t) + B \sin(2t) \). The sine term needs to be included in our guess since the derivative of the cosine term will involve sines. We solve the problem below:
3.3. **SECOND ORDER WITH CONSTANT COEFFICIENTS-UNDETERMINED COEFFICIENTS**

**Example 3.7** Find one solution to the DE
\[ y'' + 5y' + 6y = 4 \cos(2t) \]

**Solution:** We guess at the form of a solution \( y = A \cos(2t) + B \sin(2t) \).

\[ y' = -2A \sin(2t) + 2B \cos(2t) \]

\[ y'' = -4A \cos(2t) - 4B \sin(2t) \]

Plugging into the left-hand DE we obtain:
\[ -4A \cos(2t) - 4B \sin(2t) + 5(-2A \sin(2t) + 2B \cos(2t)) + 6(A \cos(2t) + B \sin(2t)) \]
\[ = (2A + 10B) \cos(2t) + (-10A + 2B) \sin(2t) \]

Matching coefficients, we want the right side to be \( 4 \cos(2t) + 0 \sin(2t) \), we obtain
\[ 2A + 10B = 4 \quad \text{and} \quad -10A + 2B = 0 \]

Eliminating, (by multiplying the first equation by 10) we obtain
\[ 10A + 100B = 40 \quad \text{and} \quad -10A + 2B = 0 \]

or
\[ 102B = 40 \]

so
\[ B = \frac{20}{51} \quad \text{and} \quad A = \frac{40}{510} = \frac{4}{51} \]

So we find that
\[ y = \frac{4}{51} \cos(2t) + \frac{20}{51} \sin(2t) \]

solves the DE. □

As you may have guessed, this method works if \( f(t) \) is a function whose derivatives have forms that do not get infinitely complicated. For instance if \( f(t) = \frac{1}{t} \), then this method will not work, since to account for all possible derivatives, our guess would need to be of the form:
\[ y = A_1 \left( \frac{1}{t} \right) + A_2 \left( \frac{1}{t^2} \right) + A_3 \left( \frac{1}{t^3} \right) + \ldots \]
Example 3.8 Find one solution to the DE

\[ y'' + 5y' + 6y = 12t^2 \]

Solution: We guess at the form of a solution \( y = At^2 + Bt + C \). We need the lower order terms since the derivatives in order to account for all possible forms of the derivatives. We compute

\[ y' = 2At + B \]
\[ y'' = 2A. \]

Plugging into the left-hand DE we obtain:

\[ 2A + 5(2At + B) + 6(At^2 + Bt + C) \]
\[ = (6A)t^2 + (10A + 6B)t + 2A + 5B + 6C \]

matching coefficients, we want the right side to be \( 1t^2 + 0t + 0 \), we obtain

\[ 6A = 12 \text{ and } 10A + 6B = 0 \text{ and } 2A + 5B + 6C = 0 \]

solving, we obtain

\[ A = 2 \text{ and } B = -\frac{10}{3} \text{ and } C = \frac{19}{9} \]

So

\[ y = 2t^2 - \frac{10}{3}t + \frac{19}{9} \]

solves the DE. □

We provide one more example to illustrate the need make a guess whose derivatives are also of the form of the original guess.

Example 3.9 Find one solution to the DE

\[ y'' + 4y' + y = 2xe^{-x} \]
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**Solution:** We guess at the form of a solution \( y = Axe^{-x} + Be^{-x} \). Again, we need our guess to have the property that the derivatives of our guess have the same form as our guess itself.

\[
y' = Ae^{-x} - Axe^{-x} - Be^{-x}
\]

\[
y'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} + Be^{-x}.
\]

Plugging into the DE, we obtain:

\[
-2Ae^{-x} + Be^{-x} + Axe^{-x} + 4(Ae^{-x} - Axe^{-x} - Be^{-x}) + Axe^{-x} + Be^{-x}
\]

\[
= -2Ae^{-x} + (2A - 2B)e^{-x},
\]

which we wish to set equal to

\[
2xe^{-x} + 0e^{-x}
\]

Matching coefficients,

\[-2A = 2 \quad \text{and} \quad 2A - 2B = 0\]

so

\[
A = -1 \quad \text{and} \quad B = -1
\]

and \( y = -xe^{-x} - e^{-x} \). □

This method could conceivably break down if the general solution to the homogeneous DE has the same form (or parts of the same form) as the guess function as demonstrated by the following example:

**Example 3.10** Show that \( y = Ae^{-5x} \) cannot be a solution to the equation

\[
y'' + 6y' + 5y = 4e^{-5x}.
\]

**Solution:** Consider \( y = Ae^{-5x} \). We know \( y' = -5Ae^{-5x} \) and \( y'' = 25Ae^{-5x} \).

Plugging into the left-hand side of the DE we obtain

\[
25Ae^{-5x} + 6(-5Ae^{-5x}) + 5(Ae^{-5x}) = 0
\]
The trouble here is that our guess actually coincides with a term from the homogeneous general solution so there is no possible choice of $A$ to satisfy the DE.

Recall that the characteristic polynomial of the DE is

$$r^2 + 6r + 5 = (r + 1)(r + 5)$$

which has roots $r_1 = -1$ and $r_2 = -5$.

So the general solution to the homogeneous is

$$y_{homo} = c_1e^{-x} + c_2e^{-5x}$$

whose second term coincides with the that guess we made. □

In this case, we can recover the solution to the DE by multiplying the original guess by $x$. (This may seem out of the blue, but when one considers the effects of the product rule, this scheme makes better sense).

**Example 3.11** Show that there is a solution of the form $y = Axe^{-5x}$ to the differential equation

$$y'' + 6y' + 5y = 4e^{-5x}.$$ 

**Solution:** We know $y' = -5Axe^{-5x} + Ae^{-5x}$ and $y'' = 25Axe^{-5x} - 5Ae^{-5x} - 5Ae^{-5x}$. Plugging into the left-hand side of the DE we obtain

$$25Axe^{-5x} - 10Ae^{-5x} + 6(-5Axe^{-5x} + Ae^{-5x}) + 5(Axe^{-5x})$$

$$= -4Ae^{-5x},$$

so if $A = -1$ then the left-side will equal the right-hand side. So $y = -xe^{-5x}$ solves the DE. □

In the previous example, note that we do not need a term of the form $Be^{-5x}$ since plugging into the DE will just result in zero as we saw before. In general practice, if a term of your guess coincides with a term from the homogeneous DE, one multiplies by the smallest power of the independent variable and adjusts the guess. In the next example, we are forced to multiply by the square of the variable.
Example 3.12 Find one to the differential equation

\[ y'' - 8y' + 16y = e^{4t}. \]

Solution: Notice that the guess

\[ y = Ae^{4t} \]

unfortunately coincides with one term in the homogeneous solution

\[ y = c_1 e^{4t} + c_2 te^{4t}. \]

So we have no hope of finding an \( A \) to solve the nonhomogeneous. To remedy, we multiply our guess by \( t^2 \) (multiplying by \( t \) will still not work).

So we will work with

\[ y = At^2 e^{4t} \]

We know \( y' = 2Ate^{4t} + 4At^2e^{4t} \) and \( y'' = 2Ae^{4t} + 8Ate^{4t} + 8Ate^{4t} + 16At^2e^{4t} \). Plugging into the left-hand side of the DE we obtain

\[ 2Ae^{4t} + 16Ate^{4t} + 16At^2e^{4t} - 8 \left( 2Ate^{4t} + 4At^2e^{4t} \right) + 16 \left( At^2e^{4t} \right) \]

\[ = 2Ae^{4t}, \]

so if \( A = \frac{1}{2} \) then the left-side will equal the right-hand side. So \( y = \frac{1}{2} t^2 e^{4t} \) solves the DE. □

Below is a chart to assist in determining the appropriate guess:
### Undetermined Coefficients Method

The nonhomogeneous second order linear DE with constant coefficients

\[ ay'' + by' + cy = f(t) \]  \hspace{1cm} (3.5)

has a solution of the form:

\[
\begin{array}{|c|c|}
\hline
f(t) & \text{Guess} \\
\hline
e^{rt} & Ae^{rt} \\
\hline
\cos(kt) & A\cos(kt) + B\sin(kt) \\
\hline
a_nt^n + a_{n-1}t^{n-1} + ... + a_1t + a_0 & A_nt^n + A_{n-1}t^{n-1} + ... + A_1t + A_0 \\
\hline
t^n \cos(kt) & \left( A_n t^n + \ldots + A_1 t + A_0 \right) \cos(kt) \\
& + \left( B_n t^n + \ldots + B_1 t + B_0 \right) \sin(kt) \\
\hline
t^n e^{rt} & \left( A_n t^n + \ldots + A_1 t + A_0 \right) e^{rt} \\
\hline
\end{array}
\]

The unknown constants in the guess are obtained by plugging the guess into the DE and solving for the coefficients.

The sole exception is in the special case when terms from the above guess coincide with the homogeneous solution to the DE. In these cases, the guess term needs to be multiplied by the smallest power of \( t \) so that the guess no longer has terms that coincide with the general solution of the homogeneous.

### 3.3.2 Finding general solutions to the nonhomogeneous

Next, we describe how we can use one solution to the nonhomogeneous differential equation to generate the general solution to the nonhomogeneous DE. The following theorem concerning linear DEs allows us to do this. This theorem is sometimes called the superposition principle, since one solution can be superimposed onto the other. Note that this theorem holds for general linear second order ODE (which includes the case of constant coefficients).
### Superposition Principle

Suppose $y_1(t)$ solves the linear DE

$$a(t)y'' + b(t)y' + c(t)y = f(t) \quad (3.6)$$

and that $y_2(t)$ solves the linear DE

$$a(t)y'' + b(t)y' + c(t)y = g(t) \quad (3.7)$$

Then $Y(t) = y_1(t) + y_2(t)$ solves

$$a(t)y'' + b(t)y' + c(t)y = f(t) + g(t) \quad (3.8)$$

**Proof:** Consider

$$Y(t) = y_1(t) + y_2(t).$$

Differentiating and using the fact that the derivative of the sum equals the sum of the derivatives:

$$Y'(t) = y'_1(t) + y'_2(t).$$

$$Y''(t) = y''_1(t) + y''_2(t).$$

So plugging into the left side of DE (3.11), we obtain

$$a(t)Y'' + b(t)Y' + c(t)Y$$

$$= a(t)(y''_1 + y''_2) + b(t)(y'_1 + y'_2) + c(t)(y_1 + y_2)$$

$$= a(t)y''_1 + b(t)y'_1 + c(t)y_1 + a(t)y''_2 + b(t)y'_2 + c(t)y_2$$

$$= f(t) + g(t).$$

So $Y(t)$ solves DE (3.11). □

Note that in the above theorem, we assume that $Y(t) = y_1(t) + y_2(t)$ exists, meaning that we assume that $y_1$ and $y_2$ have compatible domains.

The following result is a straightforward consequence of the superposition principle:
Using Undetermined Coefficients for General Solutions

Suppose $y_{homo}(t)$ is the general solution to the homogeneous linear DE

$$a(t)y'' + b(t)y' + c(t)y = 0 \quad (3.9)$$

and that $y_P(t)$ is any one solution to the nonhomogeneous linear DE

$$a(t)y'' + b(t)y' + c(t)y = f(t) \quad (3.10)$$

Then $Y(t) = y_{homo}(t) + y_P(t)$ is the general solution for

$$a(t)y'' + b(t)y' + c(t)y = f(t) \quad (3.11)$$

This allows us to find general solutions to nonhomogeneous DEs. First, we find the general solution to the associated homogeneous DE and add it to one solution to the nonhomogeneous DE.

**Example 3.13** Find the general solution to the differential equation

$$y'' - 2y' + y = 3e^{4t}.$$ 

**Solution:** As before, we can find one solution to the nonhomogeneous by making a guess

$$y = Ae^{4t}$$

Plugging into the left-side of the DE, we obtain

$$16Ae^{4t} - 8Ae^{4t} + Ae^{4t} = 9Ae^{4t}$$

So to match the right side of the DE, we take $A = \frac{1}{3}$ or we see that one solution to the nonhomogeneous DE is

$$y_P(t) = \frac{1}{3}e^{4t}$$

Next, we realize that the general solution to

$$y'' - 2y' + y = 0$$

is given by

$$y_{homo}(t) = c_1e^t + c_2te^t$$
Thus, by the superposition principle,

\[ y(t) = y_{\text{homo}}(t) + y_P(t) = c_1 e^t + c_2 te^t + \frac{1}{3} e^{4t} \]

is the general solution to the nonhomogeneous DE. □

Exercises

Find ONE solution for each of the DEs

1. \[ y'' + 4y' + 4y = 2e^t \]
2. \[ y'' + 3y' + 2y = t^2 - 4t \]
3. \[ y'' + 6y' - 7y = \cos(4t) \]
4. \[ z'' + 4z' + z = 4 \]
5. \[ z'' - z = 6e^x \]
6. \[ z'' + 2z' = 6 \sin t \]
7. \[ z'' + 6z' + 5z = 7e^{-t} \]
8. \[ z'' + 6z' + 5z = te^{-t} \]

Find general solutions for each of the DEs

9. \[ y'' + 4y' + 4y = \sin x \]
10. \[ y'' + 3y' + 2y = e^t \]
11. \[ y'' + 6y' - 5y = t^2 + 1 \]
12. \[ z'' + 4z' + z = t \sin t \]
13. \[ z'' - z = t^3 \]
14. \[ z'' + z' = 6 \]
15. \[ z'' + 7z' + 6z = e^{-t} \]