2.4 Applications (Exponential Growth/Decay)

Many things grow (decay) as fast as exponential functions. In general, if a quantity grows or decays at a rate proportional to quantity itself, then it will exhibit exponential behavior.

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<th>Exponential Growth/Decay</th>
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<td>Consider a quantity $Q$ which is known to change at a rate proportional to itself:</td>
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<td>$\frac{dQ}{dt} = rQ$</td>
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<td>($r$ is the constant of proportionality) Then</td>
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<td>$Q(t) = Ke^{rt}$, with $K = Q(0)$</td>
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Proof: Clearly if $Q(t) = Ke^{rt}$, then

$$Q'(t) = rKe^{rt} = rQ(t).$$

Plugging $t = 0$ into $Q(t) = Ke^{rt}$, we see that $Q(0) = K$. □

Note in equation (2.4) that $r$ is determined by the rate at which the quantity grows or decays. Clearly this depends upon the units with which $Q$ is measured and with which time is measured.

2.4.1 Radioactive Decay

It is well-known that radioactive materials decay at a rate proportional to the amount of material present.

Example 2.9 (Carbon Dating) Carbon-14 has a half-life of 5,730 years. A fossil is found to have 10% of its original Carbon-14. Determine the age of the fossil.

Solution: Let $Q(t)$ be the amount of Carbon-14 present in the fossil at time $t$, where $t = 0$ is the time of the fossilized animal. The half-life is clearly
related to the quantity $r$, since both concern the rate of decay. So we first determine $r$ by noticing that when $t = 5730$ there will be one-half of the original amount of Carbon-14, or

$$Q(5730) = \frac{1}{2} Q(0).$$

In particular

$$Q(0)e^{5730r} = \frac{1}{2} Q(0),$$

so

$$e^{5730r} = \frac{1}{2},$$

$$r = \frac{\ln(\frac{1}{2})}{5730} = -\frac{\ln 2}{5730}.$$

Next, we solve for the value of $t$ that yields 10% of its original Carbon-14. That is we wish to solve

$$Q(t) = .1Q(0)$$

for $t$.

$$Q(0)e^{-\frac{\ln 2}{5730}t} = \frac{1}{10} Q(0),$$

$$e^{-\frac{\ln 2}{5730}t} = \frac{1}{10},$$

$$-\frac{\ln 2}{5730} t = \ln(\frac{1}{10}),$$

$$t = \frac{5730 \ln 10}{\ln 2} \approx 19034.648 \text{ years}$$

2.4.2 Population Models

A biological population that is not subject to resource limitations will grow at a rate proportional to itself. This is entirely believable, since if the size of the population increases by a factor of $k$ then we would expect the growth rate to also increase by a factor of $k$ (roughly, the number of babies doubles if the population doubles). Hence, if $P(t)$ the population of a specific species at time $t$, then it is reasonable that $\frac{dP}{dt} = rP$. This is called a Malthusian
or exponential population model. Such populations grow exponentially, and this model works reasonably well when the population in question is not close to being constrained by a lack of resources. Clearly, if $r > 0$ the population is growing and if $r < 0$ the population is dying.

**Example 2.10 (Bacteria Counts)** Escherichia coli (E. Coli) is measured in colony forming units per milliliter (CFU/mL). In 'ideal' circumstances E. Coli in dairy milk has a doubling time of roughly 20 minutes. Pasteurized milk is Grade A if it has less than 1,000 CFU/mL. How long will it take for a gallon of milk with 1,000 CFU/mL to reach 1,000,000 CFU/mL (which is considered harmful) when left in 'ideal' circumstances?

**Solution:** Let $Q(t)$ be the CFU/mL of E. Coli at time $t$ (minutes). The doubling rate determines $r$ (just as half-life determines $r$).

In particular

$$Q(20) = 2Q(0)$$

so

$$Q(0)e^{20r} = 2Q(0)$$

or

$$r = \frac{1}{20} \ln 2.$$  

We wish to solve for $t$ in

$$Q(t) = 10^6,$$

where $Q(0) = 10^3$. Solving, we obtain

$$10^3 e^{\frac{\ln 2}{20} t} = 10^6$$

so

$$t = 20 \cdot \frac{\ln 10^3}{\ln 2} \approx 199.32 \text{ minutes}$$

or about 3 hours and 19.32 minutes. □
2.4.3 Financial Models

Money invested at interest generally grows in proportion to the amount of money that is invested (principal).

Example 2.11 (Saving for Retirement) A 25 year-old has inherited $50,000.00 and plans to invest it in an investment that pays 5% annual interest for 40 years. How much will be in the bank after 40 years?

Solution: Let $Q(t)$ be the dollar value of the investment at time $t$ (years). Then $\frac{dQ}{dt} = rQ$ and $Q(0) = 50,000$.

It is somewhat clear that the interest rate will dictate $r$, but how? After 1 year, we expect a growth of 5% so

$$Q(1) = Q(0) + (.05)Q(0)$$

or

$$Q(0)e^{r} = 1.05Q(0).$$

So,

$$r = \ln 1.05 \approx 0.04879016417$$

So, to solve the problem, we want

$$Q(40) = 50000e^{40\ln 1.05} = 50000 \cdot (1.05)^{40} = \$351,999.44 \square$$

One should always realize a mathematical model does not represent the actual quantities precisely. In truth, bacteria and investments do not grow continuously. Moreover, the actual quantities involved in population models can only be integers and the amount of dollars is at best measured to two decimal places. However, these models do an excellent job reflecting the real quantities that they model.

Exercises

Use the fact that $C^{14}$ has a half life of around 5730 years.

1. How old is a fossil with 25% of its original $C^{14}$?

2. Suppose that a researcher is confident that a fossil has somewhere between 15 – 35% of its original $C^{14}$. What is the range of possible ages of this fossil?
3. The production of the first atomic bomb also produced the byproduct of 1881 Ci (Curie) of the radioactive isotope Radium-226 which has a half life of 1600 years. This material is currently stored in Lewiston, NY (my hometown!!). Even 1 Curie of this material is extremely hazardous, but compute how long it will take for the 1881 Ci of Ra$^{226}$ to decay to 1 Curie.

4. A rabbit population doubles every 6 months. If the colony starts with 500 rabbits, how long will it take to reach 1500?

5. (a) An $5000 investment is made for 30 years at 8% annual interest. How much will the investment be worth?

   (b) Additionally, the investor plans to add $1200 each year to his investment. How much will the investor have after 40 years? [Hint: use the (linear) DE $\frac{dQ}{dt} = K + rQ$, where $r$ is determined by the interest rate, and $K$ is determined $Q(1) = $6600.]

6. A loan of $100,000 is taken out at 5% annual interest for 30 years.

   (a) Assume it is paid off at a continuous (and constant) rate $K$. Determine $K$ so that the loan is completely paid off in 30 years. [Hint: use the (linear) DE $\frac{dQ}{dt} = rQ - K$, where $r$ is determined by the interest rate, and $K$ is determined $Q(30) = 0$.]

   (b) What annual payment does this value of $K$ correspond to?