The Dot Product of Two Vectors

Vector Operations:

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

1. $\mathbf{u} + \mathbf{v} =$
2. For any scalar k, $k\mathbf{u} =$
3. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is ...

Warning!!!

Ex 1: Let $\mathbf{u} = \langle 0, 6, 5 \rangle$ and $\mathbf{v} = \langle 2, -3, 1 \rangle$. Compute the following:

(a) $\mathbf{u} \cdot \mathbf{v}$

(b) $\mathbf{u} \cdot \mathbf{u}$

Properties of the Dot Product:

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors.

1. Commutative Property:
2. Distributive Property:
3. If c is a scalar, then ...

4.

5. The most useful property of all!

Thm: Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors. The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by ...

Ex 2: What can you say about that angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ if you know that $\mathbf{u} \cdot \mathbf{v}$ is

(a) positive?

(b) negative?

(c) zero?

Def: We say that the vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if ...

Terminology for $\perp$:

- We say two *lines* are ______________________________ to one another.
- We say two *vectors* are ______________________________ to one another.
- We say a *vector* is ______________________________ to a *surface* or a *curve*.

Ex 3: Find the angle between $\mathbf{u} = \langle 0, 6, 5 \rangle$ and $\mathbf{v} = \langle 2, -3, 1 \rangle$.
Hint: These are the same vectors used in Example 1.

Def: Suppose $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. The **direction angles** α, β, and γ of $\mathbf{v}$ are ...

Ex 4: Suppose $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. How does the unit vector in the direction of $\mathbf{v}$ relate to the direction angles of $\mathbf{v}$?

Ex 5: Find the direction cosines and angles for the vector $\mathbf{v} = \langle -2, 6, 1 \rangle$.

Def: The **projection of the vector u onto the vector v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} =$$

Justification of the projection formula:

Ex 6: Find the magnitude of the force required to keep a 4000 lb car from rolling down a hill with 5° slope.

The Cross Product of Two Vectors

Vector Operations: So far, we know

1. $\mathbf{u} + \mathbf{v}$
2. $k\mathbf{u}$ for any scalar k
3. $\mathbf{u} \cdot \mathbf{v}$ and
4. $\text{proj}_{\mathbf{v}}\mathbf{u}$

Def: If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then the **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is $\ldots$

Warning!!! Unlike the dot product, the cross product

1.

2.

Ex 7: Let $\mathbf{u} = \langle -2, 5, 0 \rangle$ and $\mathbf{v} = \langle 2, -3, 1 \rangle$. Compute the following:

(a) $\mathbf{u} \times \mathbf{v}$

(b) $\mathbf{v} \times \mathbf{u}$

(c) $\mathbf{u} \times \mathbf{u}$

Algebraic Properties of the Cross Product:

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and c a scalar. Then

1. $\mathbf{u} \times \mathbf{v} =$

2. $\mathbf{u} \times \mathbf{u} =$

3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) =$

4. $c(\mathbf{u} \times \mathbf{v}) =$

5. $\mathbf{u} \times \mathbf{0} =$

6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) =$

Geometric Properties of the Cross Product:

1. $\mathbf{u} \times \mathbf{v}$ is ____________________ to both $\mathbf{u}$ and $\mathbf{v}$.

   Note: If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, then $\mathbf{u}$ and $\mathbf{v}$ define a plane. See figure.

   ____________________ and ____________________ are normal to this plane.

   The Right Hand Rule: Use it to ...

   (a) Point index finger in the direction of ...

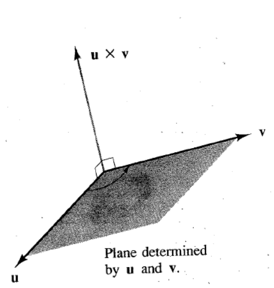
   (b) Point other fingers in the direction of ...

   (c) Then your thumb will be pointing in the direction of ...

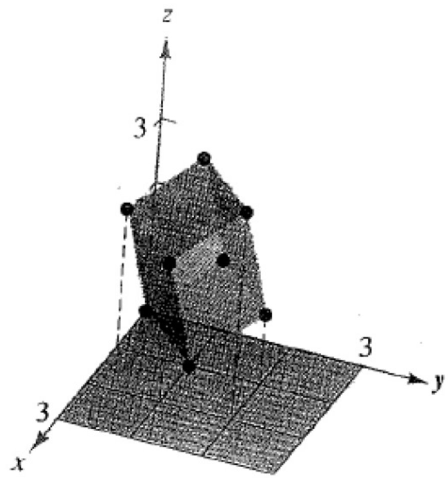
2. $\mathbf{u} \times \mathbf{v} = 0$ if and only if ...

3. The cross product $\mathbf{u} \times \mathbf{v}$ tells us the angle θ between $\mathbf{u}$ and $\mathbf{v}$ since

4. The area of the parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides is ...



Ex 2: Find the volume of the parallelpiped with vertices $(0,0,0), (1,1,0), 1,0,2), (0,1,1),$ $(2,1,2), (1,1,3), (1,2,1),$ and $(2,2,3)$.

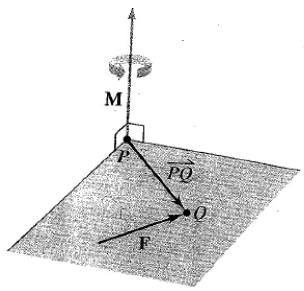


Thm: Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. Then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) =$$

Another Application of the Cross Product: In physics, the **moment** of a force, is the tendency of a force to rotate an object about an axis, fulcrum, or pivot. Just as a force is a push or a pull, the moment of a force can be thought of as a twist.

If P is the point of the axis and a force $\mathbf{F}$ is applied at the point Q on an arm $\overline{PQ}$, then the **moment** of $\mathbf{F}$ at P is given by



and the **torque** at P is given by ...

Ex 3: A child applies the brakes on a bicycle by applying a downward force of 20 pounds on the pedal when the crank makes a 40° angle with the horizontal. The crank is 6 inches in length. Find the torque at P.