Recall from Calc 1: The definition of the derivative of a function $f(x)$ is

$$f'(x) =$$

Def: The (first) partial derivative with respect to $x$ of $f(x,y)$ is defined as

$$f_x(x,y) =$$

Def: The (first) partial derivative with respect to $y$ of $f(x,y)$ is defined as

$$f_y(x,y) =$$

Notation:

- $f_x(x,y) = \frac{\partial}{\partial x}[f(x,y)] = \ldots$

- $f_y(x,y) = \frac{\partial}{\partial y}[f(x,y)] = \ldots$

Effectively, taking a partial derivative of $f(x,y)$ w.r.t. $x$ amounts to ...

Effectively, taking a partial derivative of $f(x,y)$ w.r.t. $y$ amounts to ...

Ex 1: Consider $f(x,y) = x^2 - y^2 + 2x - 3y$.

(a) $f_x(x,y)$

(b) $f_y(x,y)$
Ex 2: (An example where we MUST use the limit def.) Consider

\[
f(x, y) = \begin{cases} 
\frac{\sin(x^2 + y^2)}{x^2 + y^2} & (x, y) \neq (0, 0) \\
1 & (x, y) = (0, 0)
\end{cases}
\]

(a) Compute \( f_y(1, 1) \).

(b) Evaluate \( f_y(0, 0) \).

Ex 3: Consider \( f(x, y) = x^2 y^3 \cos(\pi x) \sin(\pi y) \).

(a) Compute \( f_y(x, y) \)

(b) Evaluate \( f_y(1, 1) \)

(c) Compute \( \frac{\partial f}{\partial x} \bigg|_{(1/2, -1/2)} \)
Geometric Interpretation of Partial Derivatives

Recall from Calc 1: Geometrically $f'(c)$ represents . . .

Similarly for Calc 3:

The figure shows the graph of a function $z = f(x, y)$. At the point $P = (x_0, y_0, f(x_0, y_0))$ we see that

- $f_x(x_0, y_0)$ represents . . .

- $f_y(x_0, y_0)$ represents . . .
Higher Order Partials

Defs:

• The second partial of $f(x, y)$ with respect to $x$ is

• The second partial of $f(x, y)$ with respect to $y$ is

• The mixed partial $f_{xy}$ is defined as . . .

• The mixed partial $f_{yx}$ is defined as . . .

Ex 4: Consider $f(x, y) = \sin(xy^2)$.
Compute $f_{xx}, f_{xy}, f_{yx}$ and $f_{yy}$
Thm: Let \( f \) be a function of two variables. If \( f_{xy} \) and \( f_{yx} \) are ... 

Interpretation of 2nd partials: As in calc 1, the second partials \( f_{xx} \) and \( f_{yy} \) tell us about ...

Section 13.6

**Directional Derivatives**

Ex 5: The figure shows the graph of a function \( z = f(x, y) \). The point \((x_0, y_0, f(x_0, y_0))\) is on this graph.

(a) How can we find the slope of the line tangent to the graph of \( f \) at \((x_0, y_0, f(x_0, y_0))\) in the \( x \)-direction?

(b) How can we find the slope of the line tangent to the graph of \( f \) at \((x_0, y_0, f(x_0, y_0))\) in the \( y \)-direction?

Ex 6: The figure shows the graph of a function \( z = f(x, y) \). The point \( P = (x_0, y_0, f(x_0, y_0)) \) is on this graph. How can we find the “slope” of the graph of \( f \) at \((x_0, y_0, f(x_0, y_0))\) in the direction of the vector \( \mathbf{v} = (1, 1) \)?
Def: Let \( z = f(x, y) \) and let \( \mathbf{v} = (a, b) \) be a vector. The \textbf{directional derivative} of \( f \) at \((x_0, y_0)\) in the direction of \( \mathbf{v} \) is given by . . .

\[ \text{Thm 1:} \] If the function \( z = f(x, y) \) is has continuous partial derivatives at and near \((x_0, y_0)\) and \( \mathbf{v} = (a, b) \) is a vector, then the directional derivative of \( f \) at \((x_0, y_0)\) in the direction of \( \mathbf{v} \) is given by . . .

(This formula is similar for functions of 3 or more variables)

\textbf{Ex 7:} Consider \( f(x, y) = xy^2 \).

(a) Find the directional derivative of \( f \) at \((-3, 1)\) in the direction of \( \mathbf{u} = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \).
(b) Find the directional derivative of $f$ at $(-3, 1)$ in the direction of $\mathbf{v} = (1, -2)$.

**Ex 8**: Consider the function $f(x, y) = x/y$. Let $P = (1, 2)$ and $Q = (7, 5)$. Find the directional derivative of $f$ at $P$ in the direction of $Q$.

**Applications of the Gradient:**

1. The direction of maximum increase of $f$ from $(x_0, y_0)$ is given by . . .

   Furthermore, the maximum value of $D_{\mathbf{u}}f(x_0, y_0)$ over all unit vectors $\mathbf{u}$ is . . .

2. The direction of maximum decrease of $f$ from $(x_0, y_0)$ is given by . . .

   Furthermore, the minimum value of $D_{\mathbf{u}}f(x_0, y_0)$ over all unit vectors $\mathbf{u}$ is . . .
3. If $\nabla f(x_0, y_0) \neq (0, 0)$ and has continuous partials at $(x_0, y_0)$ then . . .

(Similar for functions of 3 or more variables)

**Ex 9:** Let $f(x, y) = 6 - 3x^2 - y^2$.

(a) From the point $(1, 2)$, in what direction is $f$ decreasing most rapidly?

(b) How fast is $f$ decreasing in that direction?

(c) Find the level curve of $f$ containing $(1, 2)$, then sketch it along with $\nabla f(1, 2)$.

**Thm 2:** The gradient $\nabla f(x_0, y_0)$ is always orthogonal to the level curve $f(x, y) = f(x_0, y_0)$. In other words, the gradient vector is perpendicular to the level curve of $f$ through $(x_0, y_0)$. (picture)
Ex 10: Consider the function $f(x, y) = xy$.

(a) Sketch the $z = -3$ level curve.

(b) Verify that $(-1, 3)$ is on this level curve.

(c) Plot the gradient of $f$ at $(-1, 3)$ and verify that it is normal to the level curve.

Important facts

Ex 11: (An example in 3 space) Consider the function $f(x, y, z) = x^2 + y^2 + 4z^2$.

(a) Compute the gradient.

(b) Use the gradient to find the direction of maximal increase at $(2, 1, 0)$.

(c) Show that the gradient is normal to the level surface $x^2 + y^2 + 4z^2 = 5$ at $(2, 1, 0)$.