FUNCTIONS OF SEVERAL VARIABLES

A function of several variables...

The domain (unless otherwise stated) ...

Ex 1: Consider the function of two variables \( z = f(x, y) = \sqrt{16-4x^2-y^2} \). Find its domain and range and its plot its graph.

Graphing Functions of Two Variables:

For \( f(x, y) \), set \( z = f(x, y) \) and plot in 3 space.
Def: The K-level curve for a function \( z = f(x, y) \) is ...

Ex 2: Describe the level curves \( f(x, y) = 2, f(x, y) = 3, \) and \( f(x, y) = 4 \), where \( f(x, y) = \sqrt{16-4x^2-y^2} \). Then sketch the graph of \( z = f(x, y) \) in \( x, y, z \) coordinate space.
Def: A contour map for a function $f(x,y)$ is . . .

Ex 4: Sketch a contour map for $f(x,y) = \sqrt{16 - 4x^2 - y^2}$.

Ex 5: Consider the function $f(x,y) = x^2 - y^2$. Find the domain and range of $f$ and draw the level curves $f(x,y) = c$, where $c = -2, -1, 0, 1, 2$. From the contour map, make a conjecture as to the shape of the graph of $f$. 
Graphs of functions of 3 variables $w = f(x, y, z)$ are impossible since...

However, we can still construct 3-d contour plots. Instead of level curves, we obtain level surfaces.

**Ex 6:** Consider the function $f(x, y, z) = x^2 + y^2 - z$. Describe the level surfaces $f(x, y, z) = c$, where $c = -2, -1, 0, 1, 2$. 
Recall from Calc 1:

- If \( \lim_{x \to c^+} f(x) = L \) and \( \lim_{x \to c^-} f(x) = L \), then . . .

AND

- If \( \lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x) \), then . . .

Informal Def: We say \( \lim_{(x,y) \to (a,b)} f(x,y) = L \) if . . .

Note: If we want to show that \( \lim_{(x,y) \to (a,b)} f(x,y) \) Does Not Exist (DNE), then we must . . .

Ex 1: Show that \( \lim_{(x,y) \to (0,0)} \frac{7xy}{x^2 + y^2} \) does not exist.
Ex 2: Show that \( \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} \) does not exist.

Ex 3: Try to show that \( \lim_{(x,y) \to (0,0)} \frac{3x^2y}{x^2 + y^2} \) DOES exist.

Recall: To convert the point \((x, y)\) from rectangular coordinates to the polar coordinate point \((r, \theta)\), we use the formulas . . .

Observe: If \((x(t), y(t)) \to (0, 0)\), then \(r(t) \to \) ________________. Likewise, if \((x(t), y(t), z(t)) \to (0, 0, 0)\), then the spherical coordinate \(\rho(t) \to \) ________________.
Ex 4: Use polar coordinates to show that \( \lim_{(x,y) \to (0,0)} \frac{3x^2y}{x^2 + y^2} \) DOES exist.

Ex 5: Use polar coordinates to show that \( \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} \) exists.

WARNING!
Tip!!! A **shortcut** for computing limits is to use . . .

Ex 6: Use the shortcut to compute the following limits.

(a) \( \lim_{(x,y) \to (1,1)} \frac{7xy^2}{x^2 + y^2} \)

(b) \( \lim_{(x,y) \to (1,1)} \frac{7xy^2}{x^2 - y^2} \)

(c) \( \lim_{(x,y) \to (0,0)} \frac{7xy^2}{x^2 + y^2} \)

Summary:

- To show a limit DNE . . .

- To show a limit EXISTS . . .
Recall from Calc 1: We say a function $f(x)$ is **continuous** at $x = c$ if

(a)

(b)

(c)

**Def:** We say a function $f(x, y)$ is **continuous** at $(x, y) = (a, b)$ if

(a)

(b)

(c)

**Ex 7:** Determine the set of points at which the function is continuous.

(a) $f(x, y) = \begin{cases} 
\frac{3x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\
1 & \text{if } (x, y) = (0, 0)
\end{cases}$

(b) $f(x, y) = \begin{cases} 
\frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}$