Calculus of Vector-Valued Functions

HW 12.1: 69, 73, 77, 79
HW 12.2: 9, 11, 33, 43, 53, 55, 57, 61, 63, 67, 69
HW 12.3: 1, 5, 15, 19, 23, 27, 29, 31, 37

REPARAMETERIZING:

(reversing time)
$\mathbf{r}_1(t) = \langle t, t^2, 1 \rangle$
and
$\mathbf{r}_2(t) = \langle -t, (-t)^2, 1 \rangle$ trace out the same plot only ....

In general to reverse time, change all t's to ...

(Delays) To make a curve delay by K units of time, replace ....

$\mathbf{r}_1(t) = \langle t, t^2, 1 \rangle$
and
$\mathbf{r}_2(t) = \langle (t-8), (t-8)^2, 1 \rangle$ trace out the same plot only ....

Def: Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$. Then

$$\lim_{t \to a} \mathbf{r}(t) =$$

Ex 1: Let $\mathbf{r}(t) = \langle e^t, \frac{\sin t}{t}, \frac{1-\cos t}{t} \rangle$. Find $\lim_{t \to 0} \mathbf{r}(t)$.

Def: We say a vector-valued function $\mathbf{r}(t)$ is **continuous at** $t = a$ if

1.

2.

3.

We say $\mathbf{r}(t)$ is **continuous on an interval** I if ...

Ex 2: Consider $\mathbf{r}(t) = \langle e^t, \frac{\sin t}{t}, \frac{1-\cos t}{t} \rangle$.

(a) Is $\mathbf{r}(t)$ continuous when $t = 0$? Why?

(b) Can $\mathbf{r}(t)$ be extended to be continuous on the interval $(-1, 1)$?

Def: The **derivative** (velocity) of a vector-valued function $\mathbf{r}$ is defined by ...

We say $\mathbf{r}(t)$ is **differentiable at a number** c if ...

We say $\mathbf{r}(t)$ is **differentiable on an open interval** $I = (a, b)$ if ...

Notation for the Derivative of $\mathbf{r}(t)$:

Thm: If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then ...

Def: We say the curve represented by a vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is **smooth on an open interval** I if ...

1.

2.

Note:

Ex 3a: Is the curve represented by $\mathbf{r}(t) = \langle t + \sin t, 1 - \cos t \rangle$ smooth on $(-\infty, \infty)$?

Ex 3b: Is the curve represented by $\mathbf{r}(t) = \langle t^2, t^4 \rangle$ smooth on $(-\infty, \infty)$? Sketch its graph.

Thm: Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar. Then

1. (Constant Multiple Rule) $D_t[c\mathbf{r}(t)] =$

2. (Sum and Difference Rule) $D_t[c\mathbf{r}(t) \pm \mathbf{u}(t)] =$

3. (Product Rules)

(a) $D_t[f(t)\mathbf{r}(t)] =$

(b) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] =$

(c) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] =$

4. (Chain Rule) $D_t[\mathbf{r}(f(t))] =$

5. (Constant Rule)

Ex 5: Let $\mathbf{r}(t) = \langle te^t, e^t, e^t \sin t \rangle$. Compute $D_t[\mathbf{r}(t)]$ with help from an appropriate product rule.

Def: If the vector-valued function $\mathbf{r}(t)$ traces out the position of an object at time t, then the object has

- **velocity** =
- **acceleration** =
- **speed** =

Observe, when plotting tangent and acceleration vectors:

Ex 6a: Suppose $\mathbf{r}(t) = \langle t^2, t \rangle$ gives the position of an object at time t. Find the velocity, speed, and acceleration at times $t = \pm 1$ and 0 and sketch these on the curve traced out by $\mathbf{r}$.

Ex 6b: Suppose $\mathbf{r}(t) = \langle \cos t, t, \sin t \rangle$ gives the position of an object at time t. Find the velocity, speed, and acceleration at times $t = \pm\frac{\pi}{2}$ and 0 and sketch these on the curve traced out by $\mathbf{r}$.

Ex 7: An object starts from rest at the point $(0, 3, 0)$ and moves with an acceleration of $\mathbf{a}(t) = \langle 4t, 3\cos t, 3\sin t \rangle$, where $\|\mathbf{a}\|$ is measured in feet per second. Find the location of the object after $t = 2$ seconds.

When acceleration is due only to gravity:

Thm: An object with initial position $\mathbf{r}_0$, initial velocity $\mathbf{v}_0$, and a constant acceleration $\langle 0, 0, -g \rangle$ has position ...

Ex 8: Use the theorem above to solve the following:

A baseball is hit 4 feet above ground at 100 ft/sec and at an angle of $\pi/6$ with respect to the ground. Give a position function for the ball at any time t. How far will the ball go before hitting the ground?