1. INTRODUCTION. In this article we show that group representations permit simple, visual explanations of semidirect products and related concepts. Along the way, we prove an extension of Cayley’s theorem, define blocks and tails for the resulting representations, and investigate the title question: Do normal subgroups have straight tails?

We consider representations of finite groups over the field of complex numbers. Specifically, we define a representation of degree $n$ of a group $G$ to be a homomorphism from $G$ into a multiplicative group of complex $n \times n$ matrices. A more general viewpoint is certainly possible. If $F$ is any field, then an $F$-representation of $G$ is a homomorphism from $G$ into the automorphism group of some $F$-vector space.

Our first example is somewhat trivial. View the cyclic group of order three as the powers (modulo 3) of a single variable $x$: $\mathbb{Z}_3 = \langle x : x^3 = e \rangle$. We define a homomorphism $\omega$ into the multiplicative group of $3 \times 3$ complex matrices as follows:

$$
\omega(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

$$
\omega(x) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
$$

$$
\omega(x^2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.
$$

Thus $\omega$ is a representation of $\mathbb{Z}_3$ of degree three. This representation is not very efficient. It does, however, have several nice features. First, $\omega$ is faithful: the kernel of $\omega$ is the identity, so the matrix group $\omega(\mathbb{Z}_3)$ is isomorphic to $\mathbb{Z}_3$. Second, $\omega$ is a permutation representation: $\omega$ maps group elements to binary matrices with a single one in each row and column.

Cayley’s theorem implies that any finite group admits a similar representation. For the record, we state:

**Theorem 1.1 (Cayley’s Theorem).** Each group is isomorphic to some group of permutations.

For any group $G$, let $S_G$ denote the permutation group on the elements of $G$. Suppose that $G$ is finite, with $|G| = n$. Then $S_G$ is isomorphic to $S_n$, the group of permutations on $\{1, 2, \ldots, n\}$. Cayley’s theorem implies the existence of an injective homomorphism $\rho : G \rightarrow S_G$. This homomorphism identifies each element $g$ of $G$ with a permutation $\rho(g)$ of the elements. We make a fairly standard choice for $\rho(g)$; namely, the map given by $h \mapsto gh$.

Next, we identify the resulting permutations with matrices. Order the elements of $G$ as $g_1, \ldots, g_n$. For each element $g$ of $G$ we define the $n \times n$ binary matrix $M_{\rho(g)}$ as follows:

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Then $G$ is isomorphic to the multiplicative group of permutation matrices \( \{ M_{\rho(g)} : g \in G \} \). The isomorphism $\omega$ given by $g \rightarrow M_{\rho(g)}$ is called the (left-) regular representation of $G$. We saw the regular representation of $\mathbb{Z}_3$ in display (1.1). This terminology can vary: some authors call $\rho$ the regular representation of $G$, leaving the matrix interpretation $\omega$ to the reader.

Cayley’s theorem guarantees a faithful permutation representation for each finite group. It does not, however, guarantee an efficient representation. Section 2 shows that a careful application of Cayley’s theorem provides efficient representations of direct products. We call these representations “faithful blockings.” Section 3 describes a method for constructing faithful blockings of semidirect products. Section 4 proves an extension of Cayley’s theorem, showing that finite semidirect products always admit faithful blockings. In the final section, we briefly describe classroom applications, and suggest related undergraduate research projects.

2. FAITHFUL BLOCKINGS. The external direct product of groups $A$ and $B$ is the group of ordered pairs $P = \{(a, b) : a \in A, \ b \in B\}$, where multiplication is defined component-wise. We denote this by $A \oplus B$.

A group $G$ is factorizable if it contains nontrivial subgroups $N$ and $H$ such that $G = NH = \{nh : n \in N, \ h \in H\}$. We say that $G$ is a semidirect product of $N$ by $H$ if the following hold: $G = NH$, $N$ is normal in $G$, and $N \cap H = \{e\}$. This is denoted by $G = N \rtimes H$. If $G = N \rtimes H$ and $H$ is normal in $G$, then $G$ is an internal direct product of $N$ and $H$; we denote this by $G = N \times H$.

The concepts of internal and external direct product are equivalent. Suppose that $P$ is the external direct product of groups $A$ and $B$. The sets $A' = \{(a, e) : a \in A\}$ and $B' = \{(e, b) : b \in B\}$ form normal subgroups of $P$. Furthermore, $P$ factors as $P = A' B' = \{a'b' : a' \in A', b' \in B'\}$; we may view $P$ as the internal direct product $A' \times B'$. Conversely, every internal direct product is isomorphic to an external direct product. Therefore, we occasionally describe groups as “direct products” without specifying their precise constructions.

Assume that $G$ is a finite direct product, say $G = N \times H$. Cayley’s theorem promises a representation of $G$ with degree $|G| = |N| \cdot |H|$. We create a more convenient substitute, with degree $|N| + |H|$, by combining the regular representations of $N$ and $H$.

Let $\alpha$ and $\beta$ denote the regular representations of $N$ and $H$, respectively. If $g = nh$ with $n$ in $N$ and $h$ in $H$, then $\alpha(n)$ is an $|N| \times |N|$ permutation matrix, while $\beta(h)$ is an $|H| \times |H|$ permutation matrix. Our substitute for the regular representation of $G$ is the blocked representation $\Phi$:

$$\Phi(g) = \begin{bmatrix} \alpha(n) & 0 \\ 0 & \beta(h) \end{bmatrix}.$$ 

In particular,

$$\Phi(n) = \begin{bmatrix} \alpha(n) & 0 \\ 0 & I \end{bmatrix}, \quad \Phi(h) = \begin{bmatrix} I & 0 \\ 0 & \beta(h) \end{bmatrix}.$$ 

The formal statement of this idea, Corollary 2.1, is relatively easy to prove. In section 4, we show that a similar statement is true for semidirect products.
Corollary 2.1 (Cayley’s Theorem for Direct Products). If \( G \) is a finite direct product of \( N \) and \( H \), then \( G \) is isomorphic to a subgroup of \( S_N \oplus S_H \) and admits a faithful permutation representation in \( |N| \times |N| \) and \( |H| \times |H| \) blocks.

Assume that \( G \) is a finite factorizable group, \( G = NH \). We hope to construct a faithful permutation representation of \( G \) from representations of \( N \) and \( H \). A faithful blocking of \( G \) is a faithful permutation representation \( \Phi \) of the type

\[
\Phi(g) = (\alpha, \beta)(g) = \begin{bmatrix}
\alpha(g) & 0 \\
0 & \beta(g)
\end{bmatrix},
\]

where \( \alpha \) and \( \beta \) are representations of \( G \) whose restrictions to \( N \) and \( H \), respectively, are faithful.

We use the terms “blocks” and “tails” to describe the nonzero parts of faithful blockings. If \( \Phi = (\alpha, \beta) \) and \( n \) is an element of \( N \), then the block of \( n \) is \( \alpha(n) \), while the tail of \( n \) is \( \beta(n) \). Conversely, an element \( h \) of \( H \) has block \( \beta(h) \) and tail \( \alpha(h) \). An element of \( N \) or \( H \) has a straight tail if its tail is the appropriate identity matrix; if an element’s tail is not straight, it is twisted.

We extend these definitions to the subgroups \( N \) and \( H \): block \( (N) = \alpha(N) \), tail \( (N) = \beta(N) \), etc. The subgroups \( N \) and \( H \) have twisted tails if they contain elements with twisted tails; the subgroups have straight tails if they contain only elements with straight tails.

Example 2.1 exhibits a faithful blocking of a group \( G \) that is isomorphic to \( \mathbb{Z}_3 \oplus S_3 \). Blocks are distinguished in the notation by brackets, whereas tails do not have brackets. Zeroes in tails are abbreviated by dots, highlighting patterns formed by 1s.

**Example 2.1.** Faithful blocking of a direct product.

\[
G = \langle x, a, b : x^3 = a^3 = b^2 = e, xa = ax, xb = bx, ab = ba^{-1} \rangle
= \langle x : x^3 = e \rangle \times \langle a, b : a^3 = b^2 = e, ab = ba^{-1} \rangle
\cong \mathbb{Z}_3 \oplus S_3
\]

\[
\Phi(x) = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
O
1 \cdot \cdot \cdot \\
\cdot 1 \cdot \cdot \\
\cdot \cdot 1 \cdot \\
\cdot \cdot \cdot 1 \cdot \\
\cdot \cdot \cdot \cdot \cdot 
\]

\[
\Phi(a) = \begin{bmatrix}
1 & \cdot \cdot \cdot \\
\cdot 1 \cdot \\
\cdot \cdot 1 \\
\cdot \cdot \cdot 1
\end{bmatrix}
O
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
We view elements of \( \Phi(\langle x \rangle) \) as \( 3 \times 3 \) blocks with attached \( 6 \times 6 \) tails. Conversely, we view elements of \( \Phi(\langle a, b \rangle) \) as \( 6 \times 6 \) blocks with \( 3 \times 3 \) tails.

It seems reasonable to call the behavior of \( \Phi(\langle x \rangle) \) and \( \Phi(\langle a, b \rangle) \) normal. In direct products, blocks can be viewed separately; the associated tails are just identities. Can we assume that normal subgroups have straight tails? Can we assume that subgroups that are not normal have twisted tails?

**Conjecture 1 (Normal Subgroups).** Any factorizable finite group \( G = NH \) admits a faithful blocking. Furthermore, \( N \) (respectively, \( H \)) is normal in \( G \) if and only if \( N \) (respectively, \( H \)) has a straight tail under the faithful blocking.

Conjecture 1 is certainly true if \( G = N \times H \). Example 2.2 demonstrates that the conjecture is credible for the broader class of semidirect products.

Recall that \( G \) is a semidirect product of \( N \) by \( H \) if the following hold: \( G = NH \), \( N \cap H = \{ e \} \), and \( N \) is normal in \( G \). In Example 2.2, we create a faithful blocking for the semidirect product

\[
G = \langle x, y : x^3 = y^4 = e, yxy^{-1} = x^{-1} \rangle = \langle x : x^3 = e \rangle \ltimes \langle y : y^4 = e \rangle = X \ltimes Y.
\]

The generator of \( X \) must have a block with multiplicative order three. We assume the tail of \( X \) is straight. The generator of \( Y \) must have a block with order four. As \( Y \) is not normal, we suspect that tail(\( Y \)) cannot be straight. Therefore, we create a twisted tail, with multiplicative order two, for its generator \( y \). The resulting matrix group is isomorphic to \( G \). Indeed, it represents \( G \) in a faithful blocking.

**Example 2.2.** Faithful blocking of a semidirect product.

\[
G = \langle x, y : x^3 = y^4 = e, yxy^{-1} = x^{-1} \rangle = X \ltimes Y
\]

We view elements of \( \Phi(\langle x \rangle) \) as \( 3 \times 3 \) blocks with attached \( 6 \times 6 \) tails. Conversely, we view elements of \( \Phi(\langle a, b \rangle) \) as \( 6 \times 6 \) blocks with \( 3 \times 3 \) tails.
In Example 2.1 (a direct product), the tails are largely place-holders. In Example 2.2 (a semidirect product), the tails are significant. In particular, Conjecture 1 predicts that normal subgroups can be identified by their tails. We test this prediction for the subgroup $Y$.

$$
\Phi(x)\Phi(y)\Phi(x)^{-1} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
$$

The resulting matrix is not in $\Phi(Y)$, so $xYx^{-1} \not\subseteq Y$. The twisted tail of $Y$ is not preserved under conjugation, implying that $Y$ is not normal in $G$.

Before we consider the subgroup $X$, we remark that the usual normality test can be shortened for factors of a group. Since $G = XY$ and $Y = \langle y \rangle$, $X$ is normal in $G$ if and only if $yXy^{-1} \subseteq X$.

The straight tail of $X$ is preserved under conjugation, since

$$\text{block}(y) \cdot \text{tail}(x^b) \cdot \text{block}(y^{-1}) = I$$

for any integer $b$. Conjugation also preserves block$(X)$. For each integer $b$ the matrix tail$(y) \cdot \text{block}(x^b) \cdot \text{tail}(y^{-1})$ either leaves block$(x^b)$ unchanged or inverts it. We thus see that $yXy^{-1} \subseteq X$, implying that $X$ is normal in $G$.

We appear to have a characterization for semidirect products.

**Conjecture 2 (Faithful Blockings of Semidirect Products).** If $G$ is a finite semidirect product of $N$ by $H$, then $G$ admits a faithful blocking in $|N| \times |N|$ and $|H| \times |H|$ blocks. Furthermore, $N$ has a straight tail, and tails from $H$ act on blocks from $N$ by conjugation.

Suppose that our conjecture is accurate. We can then view any semidirect product $G = N \rtimes H$ as a group of blocked matrices. The group of blocks $\{\text{block}(x) : x \in N\}$ faithfully represents $N$, while the group of blocks $\{\text{block}(y) : y \in H\}$ faithfully represents $H$. The group of tails $\{\text{tail}(y) : y \in H\}$ accurately represents the interaction between the subgroups $N$ and $H$.

**3. FAITHFUL BLOCKINGS OF SEMIDIRECT PRODUCTS.** In this section, we describe a construction for faithful blockings of semidirect products. We delay until section 4 the formal proof that this construction produces faithful blockings.

Recall the faithful blocking of Example 2.2. The blocks of $X$ and $Y$ were essentially the regular representations of $X$ and $Y$. We would like to explain tails of faithful blockings in a similar manner. To do this, we require a well-known generalization of Cayley’s theorem.
The generalized Cayley theorem. Cayley’s theorem identifies each element of a group $G$ with a permutation of the elements of $G$. A more general viewpoint identifies each element of $G$ with a permutation of the cosets of $G$ relative to some subgroup.

Assume that $A$ is a subgroup of $G$, and let $G/A$ signify the collection of left cosets of $G$ with respect to $A$ (i.e., $G/A = \{gA : g \in G\}$). For each $p$ in $G$ define the coset permutation $\phi_A[p]$ to be the bijection on $G/A$ given by $gA \mapsto (pg)A$. Now $G/A$ forms a group only if $A$ is normal, but the set $S_{G/A}$ of all coset permutations is always a group under composition. (For more details, see Adkins [1].)

The resulting generalization of Cayley’s theorem paraphrases a result from Gal- lian’s book [5, p. 420]:

**Theorem 3.1 (Generalized Cayley Theorem).** If $A$ is a subgroup of $G$, then $S_{G/A}$ is a group and the mapping $\phi_A : G \rightarrow S_{G/A}$ is a homomorphism. If $M$ is a normal subgroup of $G$ that is contained in $A$, then $M \subseteq \ker(\phi_A) \subseteq A$.

Assume that a group $G$ factors as $G = NH$. Theorem 3.1 defines corresponding homomorphisms $\phi_H$ and $\phi_N$. The kernel of $\phi_H$ is the largest subgroup of $H$ that is normal in $G$ (it is frequently called the core of $H$):

$$\text{core}(H) = \ker(\phi_H) = \{ p \in G : (pg)H = gH \text{ for all } g \in G \}.$$  

Thus, Theorem 3.1 implies that $\phi_H(G) \cong G/\text{core}(H)$ and, by similar reasoning, $\phi_N(G) \cong G/\text{core}(N)$.

Cayley’s theorem and semidirect products. We earlier produced faithful blockings of direct products by mapping $G = N \times H$ into $S_N \oplus S_H$. Since $N$ is normal in $G$, $G/N \cong H$ and $S_{G/N} \cong S_H$. Similarly, $S_N$ and $S_{G/N}$ are isomorphic. We could say that, because $G = N \times H$ is isomorphic to a subgroup of $S_{G/H} \oplus S_{G/N}$, direct products admit faithful blockings.

Theorem 3.1 suggests an extension of this logic to general factorizations. If $G = NH$, then $G$ admits the homomorphisms $\phi_N : G \rightarrow S_{G/N}$ and $\phi_H : G \rightarrow S_{G/H}$. Define $\phi = (\phi_H, \phi_N)$ to be the homomorphism from $G$ into $S_{G/H} \oplus S_{G/N}$ given by $\phi[g] = (\phi_H[g], \phi_N[g])$. We are searching for faithful blockings. A natural question is: Does $\phi$ produce the faithful components we require?

From Theorem 3.1, we know that $\ker(\phi_H) \subseteq H$, while $\ker(\phi_N) \subseteq N$. Thus, $\ker(\phi) \subseteq N \cap H$, so $\phi$ is one-to-one if $N \cap H = \{e\}$. We thus obtain an extension of Cayley’s theorem:

**Proposition 3.1.** If $G = NH$ and $N \cap H = \{e\}$, then $G$ is isomorphic to a subgroup of $S_{G/H} \oplus S_{G/N}$.

A modification of $\phi$ should produce the desired matrix representations of $G = NH$. Define $\omega_H$ to be the injective homomorphism that identifies each permutation $p$ of $S_{G/H}$ with the corresponding binary matrix $M_p$. We can convert the homomorphism $\phi_H : G \rightarrow S_{G/H}$ to a matrix representation by composing it with $\omega_H$. Thus, $\Phi_H = \omega_H \circ \phi_H$ maps $G$ into $|G/H| \times |G/H|$ matrices. Similarly, we define $\omega_N$ and $\Phi_N = \omega_N \circ \phi_N$. We assemble these pieces to form $\Phi = (\Phi_H, \Phi_N)$, which maps $G$ into blocked permutation matrices. If $N \cap H = \{e\}$, then $\Phi$ is faithful.

We now have sufficient information to prove that semidirect products admit faithful blockings. Before completing the proof, however, we use the methods that we have just described to construct an example of a faithful blocking.
Construction of faithful blockings. Let $G$ be the semidirect product with presentation:

$$G = \langle x, y : x^5 = y^8 = e, yxy^{-1} = x^2 \rangle$$

$$= \langle x : x^5 = e \rangle \rtimes \langle y : y^8 = e \rangle$$

$$= N \rtimes H.$$

Note that $N$ is normal in $G$. Furthermore, $H = \langle y \rangle$ acts on $N = \langle x \rangle$ by conjugation. We expect $G$ to admit a faithful blocking in which $N$ has a straight tail. The representation should produce matrices containing $8 \times 8$ and $5 \times 5$ blocks. (This is somewhat more manageable than the regular representation, with its $40 \times 40$ matrices.)

We first deal with $\Phi_N$. For this, we order the elements of the quotient group $G/N$ as $N, yN, \ldots, y^7 N$.

We define $\phi_N$ via the left-action of the generators $y$ and $x$ on $G/N$. In other words, $\phi_N[y]$ and $\phi_N[x]$ are the permutations of $G/N$ that occur when we multiply cosets on the left by $y$ and $x$, respectively.

We begin with $\phi_N[y]$. For any exponent $k$, we see that

$$\phi_N[y](y^k N) = (yy^k)N = y^{k+1(\text{mod} 8)}N.$$

Thus, $\phi_N[y]$ cyclically permutes the collection of cosets.

Now consider the action of $x$ on $G/N$. From the group presentation, we have $yxy^{-1} = x^2$. This implies that $xy = yx^3$, whence for any $k$

$$\phi_N[x](y^k N) = (xy^k)N = y^k(x^{3k(\text{mod} 5)}N) = y^k N.$$

In other words, $\phi_N[x]$ is the identity permutation of $G/N$.

The homomorphism $\phi_N$, and the corresponding matrix representation $\Phi_N = \omega_N \circ \phi_N$, are now fully determined. The restrictions of $\phi_N$ and $\Phi_N$ to $H$ are faithful, while their restrictions to $N$ are trivial. The mapping $\Phi_N$ defines tail($x$) and block($y$) in a partially completed faithful blocking:


$$\Phi_N[y] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We might have predicted the straight tail of $x$, for $x$ belongs to $N$ and $N$ is normal in $G$.

We next turn to $\Phi_H$. Here we order the cosets that comprise $G/H$ as

$$H, xH, \ldots, x^4 H.$$

The permutations $\phi_H[x]$ and $\phi_H[y]$ are the left-actions of $x$ and $y$ on $G/H$. 

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The mapping $\phi_H[x]$ is a cyclic permutation given by $x^k H \to x^{k+1(\text{mod} 5)} H$. The action of $y$ is more complex. From the group presentation, we have $yx = x^2 y$, so

$$\phi_H[y] (x^k H) = (yx^k) H = x^{2k(\text{mod} 5)}(y H) = x^{2k(\text{mod} 5)} H$$

for any exponent $k$. We conclude that $\phi_H[y]$ fixes $H$, while exchanging the other cosets.

The corresponding matrix representation $\Phi_H = \omega_H \circ \phi_H$ takes the form:

$$\Phi_H[y] = \begin{bmatrix} 1 & . & . & . \\ . & . & 1 \\ . & . & . & 1 \\ . & 1 & . & . \end{bmatrix}, \quad \Phi_H[x] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$ 

As tail($y$) is not the identity, $H$ has a twisted tail. This is again not surprising, since $H$ is not normal in $G$. The representation $\Phi_H$ is faithful on $N$, but not on $H$. In fact, Theorem 3.1 implies that $\Phi_H(G) \cong G/\text{core}(H) = G/(y^4)$.

Finally, we put the pieces together. The representation $\Phi = (\Phi_H, \Phi_N)$ is a faithful blocking of $G$. Blocks of elements in $N$ appear in the upper left of the matrices, while blocks from $H$ appear in the lower right. To be precise, we have

$$\Phi[x] = \begin{bmatrix} \Phi_H[x] & O \\ O & \Phi_N[x] \end{bmatrix} = \begin{bmatrix} \text{block}(x) & O \\ O & \text{tail}(x) \end{bmatrix},$$

$$\Phi[y] = \begin{bmatrix} \Phi_H[y] & O \\ O & \Phi_N[y] \end{bmatrix} = \begin{bmatrix} \text{tail}(y) & O \\ O & \text{block}(y) \end{bmatrix}.$$ 

The representation is consistent with our comments following Conjecture 2: $N$ is faithfully represented by the group of blocks from $N$; $H$ is faithfully represented by its group of blocks; and the action of $H$ on $N$ is represented by the twisted tails from $H$.

We now prove that this construction produces a faithful blocking for any finite semidirect product.

### 4. EXISTENCE OF FAITHFUL BLOCKINGS

Suppose that $G$ is a semidirect product of $N$ by $H$. Then $N \cap H = \{e\}$, so the mapping $\phi = (\phi_H, \phi_N)$ is well-defined and one-to-one. We can invoke Proposition 3.1 to assert that $G$ is isomorphic to a subgroup of $S_{G/H} \oplus S_{G/N}$. Furthermore, $N$ is normal and $G/N$ is isomorphic to $H$, so $S_{G/N} \cong S_H$. This gives:

**Corollary 4.1.** If $G$ is a semidirect product of $N$ by $H$, then $G$ is isomorphic to a subgroup of $S_{G/H} \oplus S_H$.

From here, the proof that semidirect products admit faithful blockings is straightforward. We simply verify the behavior of $\Phi = (\Phi_H, \Phi_N)$.

**Theorem 4.1 (Faithful Blockings of Semidirect Products).** If $G$ is a finite semidirect product of $N$ by $H$, then $G$ admits a faithful blocking $\Phi$ under which $N$ has a straight tail. Furthermore, $H$ has a straight tail under $\Phi$ if $G$ is a direct product; otherwise, $H$ has a twisted tail.
Proof. From Corollary 4.1, we know that $\phi : N \times H \to S_{G/H} \oplus S_H$ is injective. As the homomorphisms $\omega_N$ and $\omega_H$ are also injective, the composition $\Phi = (\Phi_H, \Phi_N)$ is faithful on $G$.

Consider the restriction of $\Phi_H$ to $N$. Since $\ker(\Phi_H) \cap N = \text{core}(H) \cap N$ and $H \cap N = \{e\}$, the restriction is faithful. By a similar argument, the restriction of $\Phi_N$ to $H$ is faithful. Accordingly, $\Phi$ is a faithful blocking of $G$.

The subgroup $N$ is normal in $G$, ensuring that $N_g = gN$ for every $g$ in $G$. Assume $n$ belongs to $N$. Then $\Phi_N[n]$ is just the identity matrix, because $\phi_N[n](gN) = n(Ng) = N_g = gN$ for any coset $gN$. We conclude that $N$ has a straight tail. Similarly, if $G = N \times H$, then $H$ is normal in $G$ and has a straight tail.

Now assume that $H$ has a straight tail. For arbitrary $n$ in $N$ and $h$ in $H$ we have

$$\Phi(nhn^{-1}) = \begin{bmatrix} \text{block}(n) \cdot \text{tail}(h) \cdot \text{block}(n^{-1}) & 0 \\ \text{tail}(n) \cdot \text{block}(h) \cdot \text{tail}(n^{-1}) & O \end{bmatrix}$$

$$= \begin{bmatrix} \text{block}(n) \cdot \text{tail}(h) \cdot \text{block}(n^{-1}) & O \\ 0 & I \cdot \text{block}(h) \cdot I \end{bmatrix}.$$ 

Since $\text{tail}(h) = I$,

$$\Phi(nhn^{-1}) = \begin{bmatrix} I & O \\ O & \text{block}(h) \end{bmatrix} = \Phi(h).$$

We conclude that $H$ is normal in $G$ and that $G$ is a direct product. $\blacksquare$

5. CONCLUSION. Faithful blockings have proved useful for teaching introductory abstract algebra. Students have constructed these representations with computer algebra systems, creating and exploring models of small groups.

The advantage of this approach is that it is visual. Students can see the internal structure of a group in its blocks and tails. Normal subgroups, quotient groups, and other fundamental concepts admit simple visual explanations.

Consider a final example. We have indicated that a group $G$ is a semidirect product of $N$ by $H$ by writing $G = N \ltimes H$, but the notation $G = N \rtimes_\theta H$ is equally common. The symbol $\theta$ represents the action of $H$ on $N$. More formally, $\theta$ is a homomorphism from $H$ into the automorphism group of $N$. Faithful blockings display such mappings as matrices: the automorphism group $\theta(H)$ is isomorphic to the matrix group $\{\text{tail}(y) : y \in H\}$.

We stated two conjectures. Conjecture 2 is correct: Theorem 4.1 confirms that finite semidirect products admit faithful blockings and that their tails reflect group structure. In semidirect products, normal subgroups have straight tails. We neither proved nor disproved Conjecture 1, which asserts that every group factorization implies a faithful blocking. This conjecture would make a good subject for undergraduate research projects. We suggest several related projects:

1. We showed that semidirect products admit faithful blockings. The proof of Proposition 3.1, however, requires only $G = NH$ and $N \cap H = \{e\}$. Are there groups that admit faithful blockings, but that are not semidirect products?

2. We found that, for $G = N \rtimes H$, $\Phi_H(G) \cong G/\text{core}(H)$. For which groups does every factorization yield $G \cong G/\text{core}(H)$? In other words, which semidirect products admit only trivial faithful blockings?

3. How does the definition of faithful blockings extend to three or more blocks? Which groups admit such representations? A possible, but quite technical, answer is contained in an article by Sysak [6].
4. Erdős and Strauss [4] proposed several interpretations of the question: How Abelian is a finite group? Is there is a simple visual interpretation for this question? Is there a simple visual answer?

5. Complements of normal subgroups are not always unique. It is even possible for a group to be simultaneously a direct product of $N$ by $H$ and a semidirect product of $N$ by another subgroup $K$ [3]. Are there visual criteria for possible complements to a normal subgroup?

6. Wild [7] classifies the groups of order sixteen as cyclic extensions from groups of order eight. Which of these extensions correspond to nontrivial faithful blockings?

These questions are only a beginning. The literature on permutation representations is extensive. Dozens of authors (reference [2], for example) give partial answers to the question: Which groups admit convenient permutation representations?

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**REFERENCES**


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